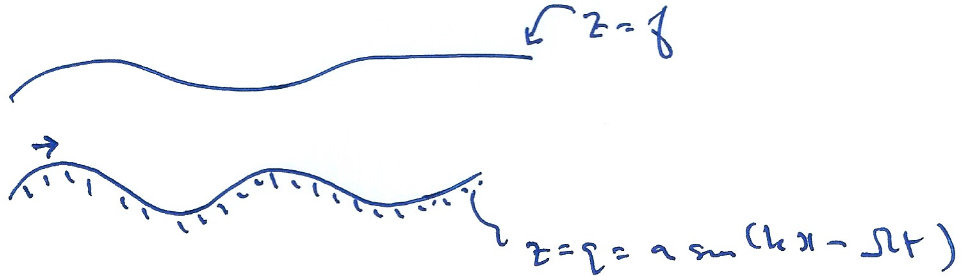
C5.7 [C6.4a]

Topics in fluid mechanics 2011 q 1

1. $\mathrm{Re}\,\dot{u} = -p_x + u_{xx} + u_{zz}$ $\dot{u} = u_t + u u_x + w u_z$ etc.

$\mathrm{Re}\,\dot{w} = -p_z + w_{xx} + w_{zz}$

$u_x + w_z = 0$



At $z = \zeta$ $\zeta_t + u\zeta_x - w = 0$

$$-p + \frac{2}{1+\zeta_x^2}\left[\zeta_x^2 u_x - \zeta_x(u_z + w_x) + w_z\right] = \hat{C}\,\frac{\zeta_{xx}}{(1+\zeta_x^2)^{3/2}}$$

$$2\zeta_x(u_x - w_z) + (\zeta_x^2 - 1)(u_z + w_x) = 0$$

(a). Each point on $z = \eta$ moves at speed $\left(\frac{\Omega}{k}, 0\right)$

hence the no-slip condition $\Rightarrow u = \frac{\Omega}{k}$, $w = 0$ on $z = \eta$

(b) $z \sim \delta$ $w \sim \delta$ $p \sim \frac{1}{\delta^2}$ $a \sim \delta$ $\eta \sim \delta$ $\zeta = \delta h$ (t the same)

In the equations $\frac{d}{dt} \to \frac{d}{dt}$ so the equations become

$$\mathrm{Re}\,\dot{u} = -\frac{1}{\delta^2}p_x + \frac{1}{\delta^2}u_{zz} + u_{xx}$$

$$\delta\,\mathrm{Re}\,\dot{w} = -\frac{1}{\delta^3}p_z + \frac{1}{\delta}w_{zz} + \delta w_{xx}$$

$$u_x + w_z = 0$$

Thus
$$\delta^2 Re\, \dot{u} = -p_x + u_{zz} + \delta^2 u_{xx}$$

$$\delta^4 Re\, \dot{w} = -p_z + \delta^2 w_{zz} + \delta^4 w_{xx}$$

$$u_x + w_z = 0$$

and at $z = h$

$$-\frac{p}{\delta^2} + \frac{2}{1+\delta^2 h_x^2}\left[\delta^2 h_x^2 u_x - \delta h_x\left\{\frac{1}{\delta} u_z + \delta w_x\right\} + w_z\right] = \frac{\hat{C}\,\delta\, h_{xx}}{\left(1+\delta^2 h_x^2\right)^{3/2}}$$

$$2\delta h_x\left(u_x - w_z\right) + \left(\delta^2 h_x^2 - 1\right)\left(\frac{1}{\delta} u_z + \delta w_x\right) = 0$$

$$h_t + u h_x - w = 0$$

So the stress conditions are

$$-p + \frac{2\delta^2}{1+\delta^2 h_x^2}\left[\delta^2 h_x^2 u_x - h_x\left\{u_z + \delta^2 w_x\right\} + w_z\right] = \frac{\hat{C}\,\delta^3 h_{xx}}{\left(1+\delta^2 h_x^2\right)^{3/2}}$$

$$2\delta^2 h_x\left(u_x - w_z\right) + \left(\delta^2 h_x^2 - 1\right)\left(u_z + \delta^2 w_x\right) = 0$$

Also at $z = q$ (or Q) $\quad u = \frac{\Omega}{h}, \; w = 0$

At leading order

$$p_x = u_{zz}$$
$$p_z = 0$$
$$u_x + w_z = 0$$

At $z = h$ $\quad h_t + u h_x - w = 0$

$$-p = C h_{xx} \qquad\qquad C = \hat{C}\delta^3$$

$$u_z = 0$$

$\Rightarrow p = -C h_{xx}$ everywhere

$$u_{zz} = -C h_{xxx}$$
$$u_z = C h_{xxx}(h - z)$$
$$u = -\tfrac{1}{2} C h_{xxx}\left[(h-z)^2 - (h-q)^2\right] + \frac{\Omega}{k}$$

$$\Rightarrow \int_q^h u\,dz = -\tfrac{1}{2} C h_{xxx}\left[-\tfrac{1}{3}(h-z)^2 - (h-q)^2 z\right]_q^h + \frac{\Omega}{k}(h-q)$$
$$= \tfrac{1}{3} C h_{xxx}(h-q)^3 + \frac{\Omega}{k}(h-q)$$

So at $z = h$: $\quad h_t + u|_h h_x - w = 0$

$$\Rightarrow h_t + u|_h h_x + \int_q^h -w_z\,dz = 0 \qquad (\text{as } w = 0 \text{ at } z = q)$$

$$\Rightarrow h_t + u|_h h_x + \int_q^h u_x\,dz = 0$$

$$= h_t + \frac{\partial}{\partial x}\int_q^h u\,dz + u|_q q_x = 0$$

$$\Rightarrow h_t + \frac{\Omega}{k} q_x + \frac{\partial}{\partial x}\left[\frac{1}{3} c h_{xxx} (h-q)^3 + \frac{\Omega}{k}(h-q)\right] = 0$$

$$\Rightarrow h_t + \frac{\Omega}{k} h_x + \frac{\partial}{\partial x}\left[\frac{1}{3} C (h-q)^3 h_{xxx}\right] = 0$$

(c) if $\frac{\Omega}{kC} \ll 1$

lead order steady state is $\frac{\partial}{\partial x}\left[\frac{1}{3} C (h-q)^3 h_{xxx}\right] = 0$

$$= (h-q)^3 h_{xxx} = \text{constant}$$

$h(x)$ (steady) but $q(x,t)$ so ~~h_{xxx}~~ $\Rightarrow$

constant $= 0$ & $h_{xxx} = 0$

$\Rightarrow h = \text{constant}$ if we assume h is bounded

(d) next if $\frac{kC}{\Omega} \ll 1$

write $\varepsilon = \frac{kC}{\Omega} \ll 1$

At lead order $h_t + \frac{\Omega}{k} h_x = 0$

so rescale $t = \frac{k}{\Omega}\tau$, then also $q = a\sin(kx - \Omega t)$

$\Rightarrow q = a\sin[k(x-\tau)]$ (rescaled)

$$\Rightarrow h_\tau + h_x + \varepsilon \frac{\partial}{\partial x}\left[\frac{1}{3}(h-q)^3 h_{xxx}\right] = 0$$

$\tau = 0$ $h = A(1 + \sin kx)$

$\Rightarrow$ approx $h = A[1 + \sin k(x-\tau)]$

& I suppose this is valid till $\tau \sim \frac{1}{\varepsilon} = \frac{\Omega}{kC}$

[question seems muddled at end: it suggests (not very clearly) that we should find the solution

for $\tau \gtrsim O\left(\frac{kC}{\Omega}\right)$ ... ah maybe it wants h correct to

$O\left(\frac{kC}{\Omega}\right) = O(\varepsilon)$: that would make better sense. So let's assume

then the question says

... hence find an approximation for h, for sufficiently small time τ, which is correct up to and including terms of $O\left(\frac{kC}{\Omega}\right)$. [You may assume $A = O(1)$.]

So we have $\quad h_\tau + h_x = -\varepsilon \frac{\partial}{\partial x}\left[\frac{1}{3}(h-q)^3 h_{xxx}\right]$

$$q = a \sin[k(x-\tau)]$$

eg write $\xi = x-\tau$, $T=\tau$, $\frac{\partial}{\partial \tau} = \frac{\partial}{\partial T} - \frac{\partial}{\partial \xi}$, $\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}$

$$\Rightarrow \frac{\partial h}{\partial T} = -\varepsilon \frac{\partial}{\partial \xi}\left[\frac{1}{3}\{h - q(\xi)\}^3 h_{\xi\xi\xi}\right]$$

$T=0$ $h = A(1+\sin k\xi)$
$q = a \sin k\xi$

$h = h_0 + \varepsilon h_1 + \ldots \quad h_0 = A(1+\sin k\xi)$

$$\frac{\partial h_1}{\partial T} = -\frac{\partial}{\partial \xi}\left[\frac{1}{3}\{A + (A-a)\sin k\xi\}^3 A \,.\, -k^3 \cos k\xi\right]$$

$$\Rightarrow h \simeq A\left[1+\sin k(x-\tau)\right] + \tau \,.\, k^4 A\left[\{A+(A-a)\sin k\xi\}^2 (A-a)\cos^2 k\xi - \frac{1}{3}\{A+(A-a)\sin k\xi\}^3 \sin k\xi\right]$$

$$= A\left[1+\sin k(x-\tau)\right] + k^4 A \tau \{A+(A-a)\sin k(x-\tau)\}^2\left[(A-a)\cos^2 k(x-\tau) - \frac{1}{3}\{A+(A-a)\sin k(x-\tau)\}\sin k(x-\tau)\right]$$

(expansion invalid at $T = O(\frac{1}{\varepsilon})$ – multiple scales.]