C5.7 Topics in fluids 2011 q 2 (old C6.4a)

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$$\rho\left[\underline{u}_t + (\underline{u}\cdot\underline{\nabla})\underline{u}\right] = -\underline{\nabla} p - \rho g \underline{k} + \mu \nabla^2 \underline{u}$$

$$\rho = \rho_0\left[1 - \alpha(T - T_0)\right]$$

(a) $x \sim d$ $\quad t \sim \dfrac{d^2}{\kappa}$, $u \sim \dfrac{\kappa}{d}$, $p + \rho_0 g z \sim \dfrac{\mu\kappa}{d^2}$

$T - T_0 \sim \Delta T$, $\rho \sim \rho_0$

(also $\rho_t + \underline{\nabla}\cdot(\rho\underline{u}) = 0$

$T_t + \underline{\nabla}\cdot(T\underline{u}) = \kappa\nabla^2 T$)

$\Rightarrow$ ~~$\alpha\Delta T\rho_0$~~ gives

non-d $\quad \rho = 1 - BT \qquad \underline{B = \alpha\Delta T}$

$$\rho_t + \underline{\nabla}\cdot(\rho\underline{u}) = 0$$

$$T_t + \underline{\nabla}\cdot(T\underline{u}) = \nabla^2 T$$

and

$$\rho_0 \frac{\kappa^2}{d^3}\rho\left[\underline{u}_t + (\underline{u}\cdot\underline{\nabla})\underline{u}\right] = -\frac{\mu\kappa}{d^3}\underline{\nabla} p + \alpha\rho_0\Delta T g T\underline{k} + \frac{\mu\kappa}{d^3}\nabla^2\underline{u}$$

$$\Rightarrow \frac{\rho_0\kappa}{\mu}\rho\left[\underline{u}_t + (\underline{u}\cdot\underline{\nabla})\underline{u}\right] = -\underline{\nabla} p + \frac{\alpha\rho_0\Delta T g d^3}{\mu\kappa} T\underline{k} + \nabla^2\underline{u}$$

or $$\frac{1}{Pr}\rho\left[\underline{u}_t + (\underline{u}\cdot\underline{\nabla})\underline{u}\right] = -\underline{\nabla} p + Ra\, T\underline{k} + \nabla^2\underline{u}$$

$$Pr = \frac{\mu}{\rho_0\kappa} \qquad Ra = \frac{\alpha\rho_0\Delta T g d^3}{\mu\kappa}$$

(b) $B \ll 1 \Rightarrow \rho = 1$

neglect $(\underline{u}\cdot\underline{\nabla})\underline{u}$ (apparently)

$$\Rightarrow \quad \underline{\nabla}\cdot\underline{u} = 0$$
$$T_t + \underline{u}\cdot\underline{\nabla}T = \nabla^2 T$$

$$\frac{1}{Pr}\underline{u}_t = -\underline{\nabla}p + Ra\,T\underline{k} + \nabla^2\underline{u}$$

take curl, $\underset{\sim}{\omega} = \text{curl}\,\underline{u} = \begin{pmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ u & v & w \end{pmatrix} = (w_y - v_z,\ u_z - w_x,\ v_x - u_y)$

$$\frac{1}{Pr}\underset{\sim}{\omega}_t = Ra\,\text{curl}(T\underline{k}) + \text{curl}\,[\text{grad}\,\underbrace{\text{div}\,\underline{u}}_{=0} - \text{curl}\,\text{curl}\,\underline{u}]$$

$$= Ra\begin{pmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & T \end{pmatrix} - \text{curl}\,\text{curl}\,\underset{\sim}{\omega}\ [+\text{grad}\,\underbrace{\text{div}\,\underset{\sim}{\omega}}_{=0}]$$

$$= Ra(T_y, -T_x, 0) + \nabla^2\underset{\sim}{\omega}$$

Now if $\zeta = \underset{\sim}{\omega}\cdot\underline{k} = v_x - u_y$

then $\zeta_t = Pr\,\nabla^2\zeta$

(c) If $u = \phi_x - \psi_y$, $v = \phi_y + \psi_x$ (∇^2 is 2-D here)

then $u_x + v_y = \nabla^2\phi$, $v_x - u_y = \nabla^2\psi$

so $w_z = -\nabla^2\phi$, $\zeta = \nabla^2\psi$

& with $\underline{u} =$ fn $\underline{u}_1(z)\exp[i(k_1x + k_2y) + \sigma t]$ (so $\phi = \Phi\exp$, $\psi = \Psi\exp$)

we have $w_1' = +k^2\Phi$, $Z = -k^2\Psi$ $\quad \zeta = Z\exp$

and $u_1 = ik_1\Phi - ik_2\Psi$

$v_1 = ik_2\Phi + ik_1\Psi$

& $\Phi = +\frac{w_1'}{K^2}$, $\Psi = -\frac{1}{K^2}Z$

$\Rightarrow \quad u_1 = +\frac{ik_1 w_1'}{K^2} + \frac{ik_2}{K^2}Z$

$v_1 = +\frac{ik_2 w_1'}{K^2} - \frac{ik_1 Z}{K^2}$

& x by $\exp[ik_1x + ik_2y + \sigma t]$

$$\Rightarrow u = \frac{1}{K^2}\left[+w_{xz} + \zeta_y\right]$$

$$v = \frac{1}{K^2}\left(w_{yz} - \zeta_x\right)$$

(d) Stress free boundaries at $z=0,1$ (non-d) $\Rightarrow \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0$

$\Rightarrow u_z = v_z = 0$ (as $w=0$)

$\Rightarrow \frac{\partial \zeta}{\partial z} = 0$

$\zeta_t = Pr\nabla^2\zeta$ here ∇^2 is 3-D

$\zeta = \eta \exp[ik_1x + ik_2y + \sigma t]$ but $\sigma \simeq 0$

so $\zeta_{zz} - K^2\zeta \simeq 0$ & $\zeta_z = 0$ on $z=0,1$

$\Rightarrow \zeta = A\cosh Kz$ can't satisfy bc^s $\Rightarrow A = 0$

so $\underline{\zeta = 0}$ [actually obvious, $\nabla^2\zeta = 0$, $\zeta = 0$ on bdns $\Rightarrow \zeta = 0$]

(e) $w = A \sin \pi z \cos k_1 x \cos k_2 y$

$\zeta = 0$

$$u = \frac{1}{K^2} w_{zx} = \frac{-A\pi k_1}{K^2} \cos \pi z \sin k_1 x \cos k_2 y$$

$$v = \frac{1}{K^2} w_{yz} = \frac{-A\pi k_2}{K^2} \cos \pi z \cos k_1 x \sin k_2 y$$

$k_1 = k_2 = k$

$$u = \frac{-A\pi k \cos \pi z}{K^2} \sin kx \cos ky$$

$$v = \frac{-A\pi k \cos \pi z}{K^2} \cos kx \sin ky$$

Streamlines in horizontal plane ~~as for~~ as for

$\cancel{u = k \sin kx \cos ky = (\sin kx \sin ky)_y}$

$\cancel{v = k \cos kx \sin ky}$

$u = \sin kx \cos ky$

$v = \cos kx \sin ky$

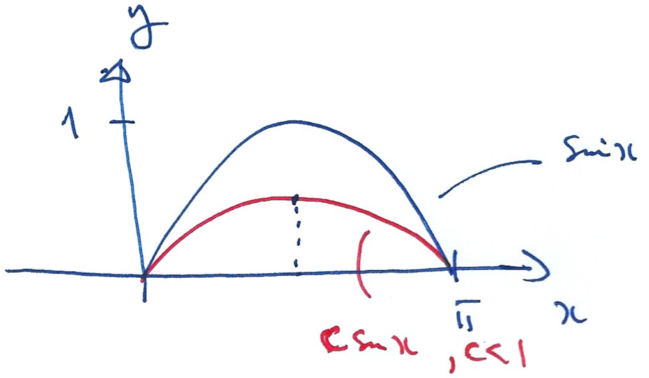
er... 'streamlines' are $\dfrac{dy}{dx} = \dfrac{v}{u} = \dfrac{\cos kx \sin ky}{\sin kx \cos ky}$

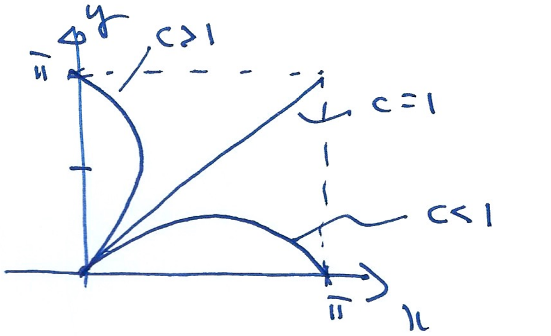
$\Rightarrow \sin ky = c \sin kx$ er?

Did he really mean this? Quite challenging I'd say

$\sin y = c \sin x$ ($k = 1$ wlog)

and in a single cell say $0 < x < \pi$, $0 < y < \pi$ (so $u = 0$ on $x = 0, \pi$; $v = 0$ on $y = 0, \pi$)





so $y \uparrow 0$ to $\sin^{-1} c$ & $\downarrow$ $x \to \pi$

if $c = 1$ $\sin x = \sin y \Rightarrow x = y$

if $c > 1$ $\sin x = \frac{1}{c} \sin y$ so just reflected

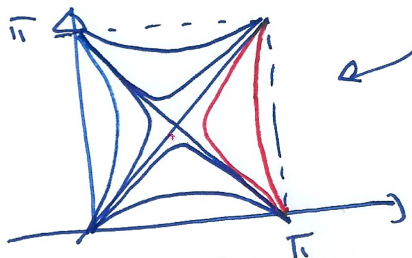
what happens if c is close to (& <) 1 ?

~~then y = x till x near π~~ — well there's a choice
$y = \sin^{-1} c \sin x \in (0, \pi/2)$
or $\pi -$ the above

~~if x = π − X~~

~~sin y = x~~

so



in red

gives projections of shear lines.
then repeated (oddly) as
same 'shearlines'
but opposite sign
of 'shear function'