C5.7 (old C6.4a) Topics in fluids 2011 q3 answer

3/

$$p_z = -\rho$$

$$Ro\,\dot{u} - v + p_x = 0$$

$$Ro\,\dot{v} + u + p_y = 0$$

$$u_x + v_y + w_z = 0$$

$$Ro\,\dot{\rho} - N^2 w = 0 \qquad \dot{\rho} = \frac{d\rho}{dt} = \rho_t + u\rho_x + v\rho_y + w\rho_z \text{ etc.}$$

(a) $Ro \ll 1$ $\quad u = u_0 + Ro\,u_1 + \dots$ etc.

lead order $\left.\begin{array}{l} v_0 = p_{0x} \\ u_0 = -p_{0y} \end{array}\right\}$ this is the geostrophic approximation

$\Rightarrow u_{0x} + v_{0y} = 0$

$\Rightarrow w_z = 0 \Rightarrow w_0 \equiv 0$ (if no flow at base)

$\Rightarrow \dot{\rho} = 0$

$u_{0x} + v_{0y} = 0 \Rightarrow \underline{u_0 = -\psi_y \quad v_0 = +\psi_x}$ (or $(\psi_y, -\psi_x)$ would do)

At next order $\quad p_{1z} = -\rho_1$

$$\frac{D_0 u_0}{Dt} - v_1 + p_{1x} = 0 \qquad \frac{D_0}{Dt} = \frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x} + v_0\frac{\partial}{\partial y}$$

$$\frac{D_0 v_0}{Dt} + u_1 + p_{1y} = 0 \qquad (\text{as } w_0 = 0)$$

$$u_{1x} + v_{1y} + w_{1z} = 0$$

$$\frac{D_0 \rho_0}{Dt} = N^2 w_1$$

(b) Eliminate p_1

$$\Rightarrow \frac{\partial}{\partial y}\left[u_{0t} + u_0 u_{0x} + v_0 u_{0y}\right] - \frac{\partial v_1}{\partial y} = -p_{1xy}$$

$$= \frac{\partial}{\partial x}\left[v_{0t} + u_0 v_{0x} + v_0 v_{0y}\right] + \frac{\partial u_1}{\partial x}$$

So
$$\frac{D_0}{Dt}\left[v_{0x}\right] + u_{0x} v_{0x} + v_{0x} v_{0y} + \frac{\partial u_1}{\partial x}$$

$$- \frac{D_0}{Dt}\left[u_{0y}\right] - u_{0y} u_{0x} - v_{0y} u_{0y} + \frac{\partial v_1}{\partial y} = 0$$

$$\Rightarrow \frac{D_0}{Dt}\left[\psi_{xx} + \psi_{yy}\right] + v_{0x}(u_{0x} + v_{0y}) - u_{0y}(u_{0x} + v_{0y})$$

$$- \frac{\partial w_1}{\partial z} = 0$$

$$\Rightarrow \frac{D_0}{Dt}(\psi_{xx} + \psi_{yy}) - \frac{1}{N^2}\frac{D_0 \rho_0}{Dt} = 0$$

So $\dfrac{D_0 q}{Dt} = 0$ where $q = \psi_{xx} + \psi_{yy} - \dfrac{\rho_0}{N^2}$

Note $\rho_0 = -p_{0z}$ and $u_0 = -\psi_y = -p_{0y}$, $v_0 = \psi_x = p_{0x}$

so $\psi = p_0$

so $\rho_0 = -\psi_{zz}$

& so $q = \psi_{xx} + \psi_{yy} + \frac{1}{N^2}\psi_{zz}$

(5) $\psi = [x^2+y^2+(Nz+1)^2]^{-1/2}$

(i) $\frac{1}{\psi^2} = x^2+y^2+(Nz+1)^2$

$$-\frac{2\psi_x}{\psi^3} = 2x, \quad -\frac{2\psi_y}{\psi^3} = 2y, \quad -\frac{2\psi_z}{\psi^3} = 2(Nz+1)$$

$$\Rightarrow \begin{aligned} \psi_x &= -x\psi^3 \\ \psi_y &= -y\psi^3 \\ \psi_z &= -(Nz+1)\psi^3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \psi_{xx} &= -\psi^3 - 3x\psi^2\psi_x = -\psi^3 + 3x^2\psi^5 \\ \psi_{yy} &= -\psi^3 + 3y^2\psi^5 \\ \psi_{zz} &= -N\psi^3 - 3(Nz+1)\psi^2\psi_z \\ &= -N\psi^3 + 3(Nz+1)^2\psi^5 \end{aligned}$$

so
$$\begin{aligned} q = &-\psi^3 + 3x^2\psi^5 \\ &-\psi^3 + 3y^2\psi^5 \\ &-\frac{1}{N}\psi^3 + \frac{3(Nz+1)^2}{N^2}\psi^5 \end{aligned}$$

$$= -\left(2+\frac{1}{N}\right)\psi^3 + 3\left\{x^2+y^2+\frac{(Nz+1)^2}{N^2}\right\}\psi^5$$

$$= \frac{-\left(2+\frac{1}{N}\right) + 3\left\{\frac{x^2+y^2+\frac{(Nz+1)^2}{N^2}}{x^2+y^2+(Nz+1)^2}\right\}}{[x^2+y^2+(Nz+1)^2]^{3/2}}$$

ii geostrophic density is $\rho_0 = -\psi_{zz}$

so $\rho_0|_{z=0} = [-N\psi^3 + 3(Nz+1)^2\psi^5]|_{z=0}$, $\psi|_{z=0} = [1+x^2+y^2]^{-1/2}$

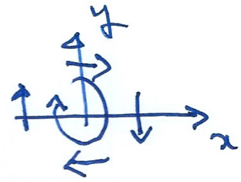
$$\Rightarrow \rho_0|_{z=0} = -N\psi_0^3 + 3\psi_0^5 = \frac{-N(1+x^2+y^2)+3}{[1+x^2+y^2]^{5/2}}$$

Flow in the atmosphere above a cold surface temperature anomaly

~~? Not sure what is meant here. I suppose cold surface $\Rightarrow$ $p_1 > 0$ at $z=0$ $\Rightarrow$ $\frac{\partial p_1}{\partial z} < 0$ at surface corresponding to increased p_1 at surface ??~~

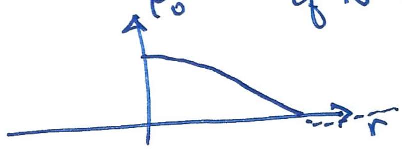
Basic flow is $u_0 = -\psi_y = y\psi^3$

$v_0 = \psi_x = -x\psi^3$



Ah. I think what is meant is this:

$$p_0\big|_{z=0} = \frac{3 - N(1+r^2)}{(1+r^2)^{5/2}}$$



if $N < 3$

so the given ψ is associated with high p_0 i.e. local surface cold air anomaly & the corresponding velocity field is

which is anticyclonic (in the Northern hemisphere)