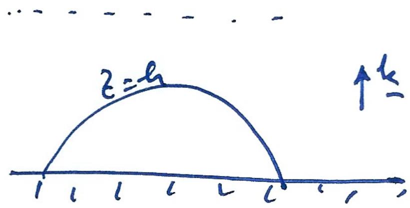
eS.7 [C6.4a]

Topics in fluids 2012 q2

1

2/ (a)



(i) $\underset{\sim}{u} = -\frac{k}{\mu}\left[\underset{\sim}{\nabla} p + \rho g \underset{\sim}{k}\right]$

(ii) $p \simeq$ hydrostatic $= \rho g(h-z)$ if flow is shallow

Then horizontal velocity is

$$\underset{\sim}{u}_H = -\frac{k}{\mu}\underset{\sim}{\nabla}_H \rho g h = -\frac{k\rho g}{\mu}\underset{\sim}{\nabla}_H h$$

Axisymmetric so $\underset{\sim}{u}_H = u\underset{\sim}{e}_r$

& $$u = -\frac{k\rho g}{\mu}\frac{\partial h}{\partial r}$$

(iii) Conservation of mass

$$\phi h_t = -\underset{\sim}{\nabla}_H \cdot (h\underset{\sim}{u}_H) \quad \text{use } \int_0^h \underset{\sim}{u}_H \, dz = h\underset{\sim}{u}_H$$

$$= -\frac{1}{r}\frac{\partial}{\partial r}(rhu)$$

So $$h_t = \frac{k\rho g}{\mu\phi}\frac{1}{r}\frac{\partial}{\partial r}(rhh_r)$$

& volume of blob is $V = 2\pi\phi\int_0^{a(t)} rh\,dr$

(iv) So let's make it non-dimensional

take $r \sim l$ (will be arbitrary as similarity solution

~~mass ⇒ V =~~ $h \sim d$, $t \sim T$, $a \sim l$

mass ⇒ $V = 2\pi\phi\, l^2 d$ ⇒ $d = \dfrac{V}{2\pi\phi l^2}$

eq ⇒ $\dfrac{d}{T} = \cancel{kg} \dfrac{k\rho g}{\mu\phi}\dfrac{d^2}{l^2}$

$$\Rightarrow T = \frac{\mu\phi l^2}{k\rho g d} = \frac{2\pi\mu\phi^2 l^4}{k\rho g V}$$

$$\Rightarrow \left.\begin{aligned} h_t &= \frac{1}{r}\frac{\partial}{\partial r}(r h h_r) \\ 1 &= \int_0^a r h \, dr \end{aligned}\right\} \text{ non-d}$$

Scaling argument $\dfrac{h}{t} \sim \dfrac{h^2}{r^2}$, $1 \sim r^2 h$

$$\Rightarrow h \sim \frac{1}{r^2} \sim \frac{r^2}{t} \Rightarrow r \sim t^{1/4} \quad h \sim t^{-1/2}$$

$$\Rightarrow a = \alpha t^{1/4} \qquad h = \frac{1}{\sqrt{t}} f(\eta) \qquad \eta = \frac{r}{t^{1/4}}$$

$$\Rightarrow -\tfrac{1}{2} t^{-3/2} f - \tfrac{1}{4} t^{-3/2} \eta f' = t^{-1/2} \frac{1}{\eta}\frac{d}{d\eta}\left[t^{-1} \eta f f'\right]$$

$$\Rightarrow -\left[\tfrac{1}{2} f + \tfrac{1}{4}\eta f'\right]\eta = (\eta f f')'$$

$$\Rightarrow (\eta f f')' + \tfrac{1}{4}\eta^2 f' + \tfrac{1}{2}\eta f = 0$$

And the $1 = \displaystyle\int_0^\alpha \eta f \, d\eta$

(v) 1st integral

$$\eta f f' + \tfrac14 \eta^2 f = \text{Constant} = 0 \quad \text{as } f = 0 \text{ at } \infty$$

$$\Rightarrow f = 0 \quad \underline{\text{or}} \quad \eta f' + \tfrac14 \eta^2 = 0$$

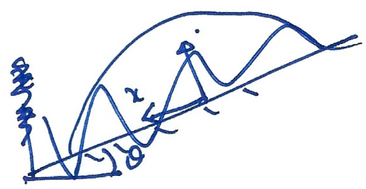
$$f' = -\cancel{\tfrac{4}{8}}\tfrac14 \eta$$

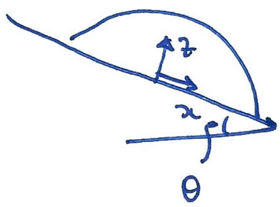
$$\underline{f = \tfrac18(\cancel{\tfrac{4}{8}}\alpha^2 - \eta^2)}$$

$$\& \quad 1 = \int_0^\alpha \tfrac18(\alpha^2\eta - \eta^3)\,d\eta$$

$$= \tfrac18\alpha^4\left(\tfrac12 - \tfrac14\right) = \frac{\alpha^4}{32} \Rightarrow \underline{\alpha = 2^{5/4}}$$

(b)





$$h_t = \frac{k\rho g}{\mu\phi}\left[\cos\theta\, \underline{\nabla}\cdot(h\underline{\nabla} h) - \sin\theta \frac{\partial h}{\partial x}\right] \qquad \underline{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

(i) $V = \dfrac{k\rho g \sin\theta}{\mu\phi}$ is the uniform rate of ~~movement~~ movement of the fluid down slope due to gravity

$$h_t + V h_x = \frac{k\rho g \cos\theta}{\mu\phi}\underline{\nabla}\cdot(h\underline{\nabla} h)$$

Define $\xi = x - Vt$, $t = \tau$ $\qquad \dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial \xi}$, $\dfrac{\partial}{\partial t} = \dfrac{\partial}{\partial \tau} - V\dfrac{\partial}{\partial \xi}$

$$\Rightarrow h_\tau = \frac{k\rho g\cos\theta}{\mu\phi}\underline{\nabla}\cdot[h\underline{\nabla} h] \qquad \underline{\nabla} = \left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial y}\right)$$

Solution of part (a). $\left[\, h_t = \frac{1}{r}\frac{\partial}{\partial r}\left(r h h_r\right),\ \int_0^a r h \, dr = 1 \right]$

(non-d) $h = \frac{1}{\sqrt{t}}\,\frac{1}{8}\left(\alpha^2 - \frac{r^2}{t^{1/2}}\right)$

re-dimensionalise,

solution of $h_t = \frac{k\rho g}{\mu\phi}\,\frac{1}{r}\frac{\partial}{\partial r}\left(r h \frac{\partial h}{\partial r}\right)$

& $V = 2\pi\phi\int_0^a r h \, dr$

is (dimensional) $\frac{a}{l} = \alpha\left(\frac{t}{T}\right)^{1/4}$, $\frac{h}{d} = \sqrt{\frac{T}{t}}\,\frac{1}{8}\left(\alpha^2 - \frac{r^2}{l^2}\frac{T^{1/2}}{t^{1/2}}\right)$

& we have to solve $h_T = \frac{k\rho g'}{\mu\phi}\,\underline{\nabla}\cdot\left[h\underline{\nabla} h\right]$, V = same

with $\underline{\nabla} = \left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial y}\right)$

So ~~s~~ solution ~~s~~ is (with $r^2 = \xi^2 + y^2$), replace g by $g' = g\cos\theta$

$$\frac{h}{d} = \frac{\sqrt{T}}{\sqrt{t}}\cdot\frac{1}{8}\left[\alpha^2 - \frac{(\xi^2 + y^2)\sqrt{T}}{l^2 t^{1/2}}\right]$$

where $T = \frac{2\pi\mu\phi^2 l^4}{k\rho g' V}$, $d = \frac{V}{2\pi\phi l^2}$ ~~so $T = \frac{\mu\phi l^2}{k\rho g d}$~~

$$\Rightarrow h = \frac{V}{2\pi\phi l^2}\,\frac{\sqrt{2\pi\mu}\,\phi\, l^2}{(k\rho g' V)^{1/2}}\,\frac{1}{8\sqrt{t}}\left[\alpha^2 - \frac{\sqrt{2\pi\mu}\,\phi\, l^2}{(k\rho g' V)^{1/2} l^2}\,\frac{\{(x - Ut)^2 + y^2\}}{t^{1/2}}\right]$$

$$= \left(\frac{\mu V}{k\rho g'}\right)^{1/2}\frac{1}{\sqrt{2\pi}}\,\frac{1}{8\sqrt{t}}\left[\alpha^2 - \sqrt{2\pi}\left(\frac{\mu\phi^2}{k\rho g' V}\right)^{1/2}\frac{\{(x - Ut)^2 + y^2\}}{t^{1/2}}\right]$$

$$= \left(\frac{\mu V}{k\rho g'}\right)^{1/2}\frac{1}{8\sqrt{t}}\left[\frac{4}{\sqrt{\pi}}\overset{=1}{\left(\frac{\alpha^2}{4\sqrt{2}}\right)} - \left(\frac{\mu\phi^2}{k\rho g' V}\right)^{1/2}\frac{\{(x - Ut)^2 + y^2\}}{\sqrt{t}}\right]$$ as required as $\alpha = 2^{5/4}$