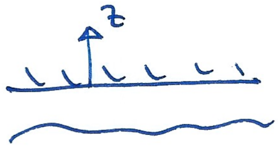
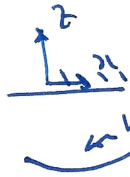
C5.7 [C6.4a] Topics in fluids 2014 q1 (1

1 (a)  $z=-h$

 $\leftarrow$ high p if $-h_{xx}>0$

so [p]

$p - p_a = -\gamma h_{xx}$

$p_x = u_{zz}$

$p_z = -1$

$z=-h \quad p = -\gamma h_{xx} \quad (p_a = 0 \text{ wlog})$

(i) $p = (-z-h) - \gamma h_{xx}$

$\Rightarrow p_x = -h_x - \gamma h_{xxx} = u_{zz}$

$\Rightarrow u_z = p_x(+h+z) \qquad (u_z = 0 \quad z=-h)$

$u = p_x(hz + \tfrac{1}{2}z^2) \qquad (u = 0 \quad z = 0)$

(ii) mass conservation

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_{-h}^{0} u\,dz = 0$$

$$\int_{-h}^{0} u\,dz = p_x\left[\tfrac{1}{2}hz^2 + \tfrac{1}{6}z^3\right]_{-h}^{0}$$

$$= p_x\left[-\tfrac{1}{2}h^3 + \tfrac{1}{6}h^3\right] = -p_x\tfrac{1}{3}h^3$$

so $$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left[-\int_{-h}^{0} u\,dz\right] = \frac{\partial}{\partial x}\left(\tfrac{1}{3}h^3 p_x\right)$$

$$= -\frac{\partial}{\partial x}\left[\tfrac{1}{3}h^3\left\{h_x + \gamma h_{xxx}\right\}\right]$$

(b) $\gamma = 1$

$$h_t = -\frac{\partial}{\partial x}\left[\tfrac{1}{3}h^3\{h_x + \gamma h_{xxx}\}\right]$$

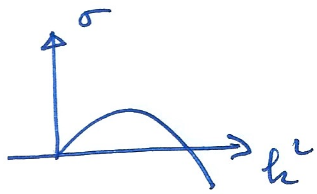
linearise $h = h_0 + \eta$

$$\Rightarrow \eta_t = -\tfrac{1}{3}\frac{\partial}{\partial x}\left[\eta_x + \eta_{xxx}\right]$$

$$\eta = e^{\sigma t + ikx}$$

$$\Rightarrow \sigma = -\tfrac{1}{3}\left[-k^2 + k^4\right]$$

ie $\sigma = \tfrac{1}{3}(k^2 - k^4)$



unstable for $k<1$

ie $\frac{2\pi}{k} > \underline{\underline{2\pi}}$ smallest unstable wavelength

most unstable max σ at $1 - 2k^2 = 0$ $\left(\frac{d}{dk^2}\right)$

$\Rightarrow k = \frac{1}{\sqrt{2}}$, $\Rightarrow \frac{2\pi}{k} = \underline{\pi\sqrt{2}}$ most unstable wavelength

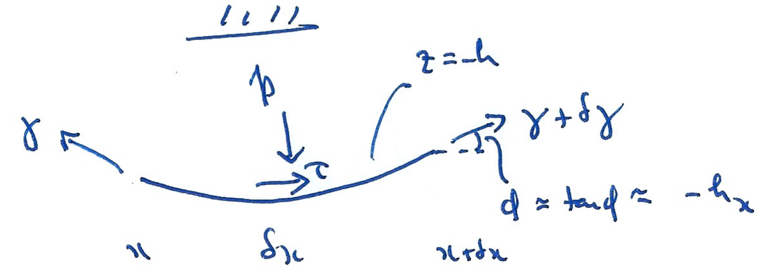
(c) Next Marangoni!

$$T = T_0 - \Delta T\frac{z}{h_0}$$

$\gamma = 1 - \frac{M h_0}{\Delta T}(T_{int} - T_0)$ & $T_{int} - T_0 = -\frac{\Delta T}{h_0}\cdot -h = \Delta T\frac{h}{h_0}$

$\Rightarrow \gamma = 1 - \frac{M h_0}{\Delta T}\cdot\frac{\Delta T h}{h_0} = 1 - Mh$ M = Marangoni number!

(i) The simplest explanation of the Marangoni effect uses the idea of surface tension as a force in the fluid. In 2-D



The normal force is $\approx -p\,\delta x + \gamma\left[-h_x\right]_x^{x+\delta x} = 0$

$$\Rightarrow \underline{p = -\gamma h_{xx}} \quad \text{as earlier on } z=-h$$

The tangential force is $\tau\,\delta x + \delta\gamma \approx 0$

$$\Rightarrow \tau = \frac{\partial u}{\partial z} \;(\text{here, non-d}) = -\frac{\partial \gamma}{\partial x}$$

then

(ii) So $u_{zz} = p_x$ as before $\quad p = (-z-h) - \gamma h_{xx}$, $\quad p_x = -h_x - (\gamma h_{xx})_x$

$$u_z = p_x(h+z) - \gamma_x$$

And $\gamma = 1 - Mh$

$$\Rightarrow u_z = p_x(h+z) + Mh_x$$

$$u = p_x\left(hz + \tfrac{1}{2}z^2\right) + Mzh_x$$

$$\int_{-h}^{0} u\,dz = -\tfrac{1}{3}h^3 p_x - \tfrac{1}{2}Mh^2 h_x$$

$$\Rightarrow h_t = \frac{\partial}{\partial x}\left[\tfrac{1}{3}h^3 p_x + \tfrac{1}{2}Mh^2 h_x\right]$$

$$= \frac{\partial}{\partial x}\left[-\tfrac{1}{3}h^3\left\{h_x + (\gamma h_{xx})_x\right\} + \tfrac{1}{2}Mh^2 h_x\right]$$

$$(\text{as } \gamma = 1 - Mh)$$

or writing it all out

$$h_t = -\frac{\partial}{\partial x}\left[\tfrac{1}{3}h^3\left\{h_x + [(1-Mh)h_{xx}]_x\right\} - \tfrac{1}{2}Mh^2h_x\right]$$

$$= -\frac{\partial}{\partial x}\left[\tfrac{1}{3}h^3\left\{h_x + (1-Mh)h_{xxx} - Mh_xh_{xx}\right\} - \tfrac{1}{2}Mh^2h_x\right]$$

as requested

iii linearised eq $h = h_0 + f$

$$f_t = -\frac{\partial}{\partial x}\left[\frac{h_0^3}{3}\left\{f_x + (1-Mh_0)f_{xxx}\right\} - \tfrac{1}{2}Mh_0^2f_x\right]$$

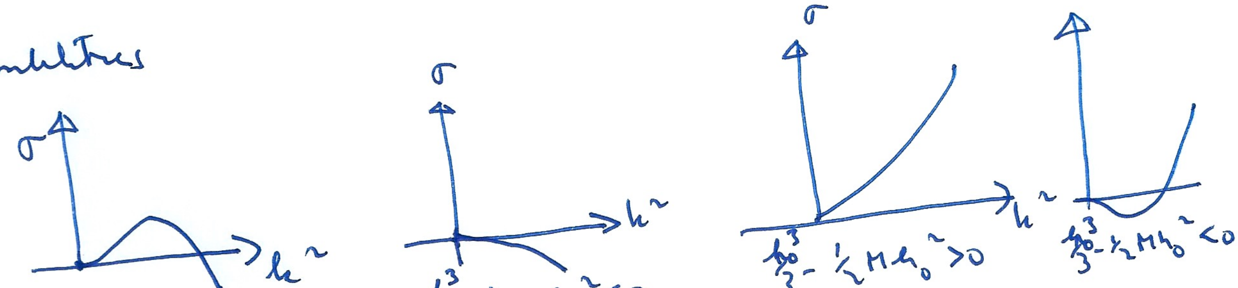
$$\sigma = -ik\left[\frac{h_0^3}{3}\left\{ik - ik^3(1-Mh_0)\right\} - \tfrac{1}{2}ikMh_0^2\right]$$

take ik out

$$= k^2\left[\tfrac{1}{3}h_0^3\left\{1-(1-Mh_0)k^2\right\} - \tfrac{1}{2}Mh_0^2\right]$$

$$= k^2\left[\frac{h_0^3}{3} - \tfrac{1}{2}Mh_0^2 - \frac{h_0^3}{3}(1-Mh_0)k^2\right]$$

possibilities



$\frac{h_0^3}{3} - \frac12 Mh_0^2 > 0$, $1-Mh_0 > 0$

$\frac{h_0^3}{3} - \frac12 Mh_0^2 < 0$, $1-Mh_0 > 0$

$\frac{h_0^3}{3} - \frac12 Mh_0^2 > 0$, $1-Mh_0 < 0$

$\frac{h_0^3}{3} - \frac12 Mh_0^2 < 0$, $1-Mh_0 < 0$

$\Downarrow$

stable $\forall\, k$ iff $\frac{h_0^3}{3} - \frac12 Mh_0^2 < 0$ & $1-Mh_0 > 0$

i.e.

$Mh_0 < 1$ and $\frac12 M > \frac{h_0}{3}$ iff $\frac{2h_0}{3} < M < \frac{1}{h_0}$

(which requires $h_0^2 < \frac{3}{2}$)