C5.7 (C6.4a) Topics in fluids 2014 q 2 (1

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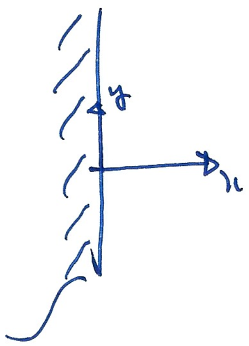

presume $\underline{g}$ is vertically downwards

$\downarrow$

$\rightarrow T \to T_\infty$

$T = T_w$

$$\underline{u} = -\underline{\nabla} p - \frac{Ra}{\alpha} \rho \underset{\sim}{e_y}$$

a bit weird with the α

rather jumbled up

$$T_t + \underline{u}\cdot\underline{\nabla} T = \nabla^2 T$$

$$\rho = 1 - \alpha(T - T_\infty)$$

(a) ~~Presume wall $T = T_\infty + \Delta T$~~ so wall $T \to 0$ at $x = \infty$

~~Well we need Bousse~~

(i) $$\underline{u} = -\underline{\nabla} p - Ra\left[\frac{1}{\alpha} - (T - T_\infty)\right] \underset{\sim}{e_y}$$

$$\psi_y = -p_x$$

$$-\psi_x = -p_y - Ra\left[\frac{1}{\alpha} - (T - T_\infty)\right]$$

$\frac{\partial}{\partial y}\frac{\partial}{\partial x} \Rightarrow \nabla^2 \psi = -Ra\, T_x$ $\qquad \psi = 0 \quad x = 0 \;\&\; x \to \infty$

$T_t + \psi_y T_x - \psi_x T_y = \nabla^2 T$ $\qquad T = T_w \quad x = 0, \; T \to T_\infty \quad x \to \infty$

(ii) thermal bl $\Rightarrow \frac{\partial}{\partial x} \gg \frac{\partial}{\partial y} \Rightarrow$ $\psi_y T_x - \psi_x T_y \simeq T_{xx}$

$$\psi_{xx} = -Ra\, T_x$$

if we lose the $\frac{\partial}{\partial t}$ also.

Hence $\psi_x = -Ra(T-T_\infty)$ with the b^c at ∞

(b) $T_w = T_\infty + y^\lambda \quad y \geq 0, x = 0$

$$\psi_y T_x - \psi_x T_y = T_{xx}$$

$$\psi_x = -Ra(T-T_\infty)$$

$$\psi = Ra^{1/2} y^{\frac{1}{2}(\lambda+1)} f(\eta) \qquad \eta = Ra^{1/2} x y^{\frac{1}{2}(\lambda-1)}$$

$$T - T_\infty = y^\lambda \Theta(\eta)$$

$$\Rightarrow \Theta(0) = 1$$
$$\Theta(\infty) = 0$$
$$f(0) = 0$$

Note T eq^n is $\frac{\partial}{\partial x}(\psi_y T) - \frac{\partial}{\partial y}(\psi_x T) = T_{xx}$

$$\text{or } \frac{\partial}{\partial x}\left[\psi_y (T-T_\infty)\right] - \frac{\partial}{\partial y}\left[\psi_x (T-T_\infty)\right] = T_{xx}$$

$$\psi_x = Ra\, y^\lambda f'$$

$$\psi_y = Ra^{1/2} \tfrac{1}{2}(\lambda+1) y^{\frac{1}{2}(\lambda-1)} f + Ra^{1/2} \tfrac{1}{2}(\lambda-1) y^{\frac{1}{2}(\lambda-1)} \eta f'$$

$$= \tfrac{1}{2} Ra^{1/2} y^{\frac{1}{2}(\lambda-1)} \left[(\lambda+1) f + (\lambda-1)\eta f'\right]$$

so $\frac{\partial}{\partial x}\left[\frac{1}{2} Ra^{1/2} y^{\frac{1}{2}(\lambda-1)} \{(\lambda+1) f + (\lambda-1)\eta f'\} y^\lambda \Theta\right]$

$$- \frac{\partial}{\partial y}\left[Ra\, y^\lambda f' y^\lambda \Theta\right] = y^\lambda Ra\, y^{\lambda-1} \Theta''$$

ugh this is nasty ...

$$\frac{\partial}{\partial x} = Ra^{\frac{1}{2}} y^{\frac{1}{2}(\lambda-1)} \frac{d}{d\eta} \quad \text{so}$$

$$\frac{1}{2} Ra\, y^{2\lambda-1} \frac{d}{d\eta}\left[\{(\lambda+1)f + (\lambda-1)\eta f'\}\Theta\right]$$

$$- Ra\left[2\lambda y^{2\lambda-1} f'\Theta + \tfrac{1}{2}(\lambda-1)\eta\, y^{2\lambda-1}(f'\Theta)'\right]$$

$$= Ra\, y^{2\lambda-1}\Theta''$$

$\div Ra\, y^{2\lambda-1}$

$\Rightarrow$

$$\frac{1}{2}\left[(\lambda+1)f\Theta + (\lambda-1)\eta\Theta f'\right]'$$

$$- \left[2\lambda f'\Theta + \tfrac{1}{2}(\lambda-1)\eta(\Theta f')'\right] = \Theta''$$

$$\Rightarrow \tfrac{1}{2}(\lambda+1)(f\Theta)' + \tfrac{1}{2}(\lambda-1)\Theta f' + \tfrac{1}{2}(\lambda-1)\eta(\Theta f')'$$

$$- 2\lambda\Theta f' - \tfrac{1}{2}(\lambda-1)\eta(\Theta f')' = \Theta''$$

$$\Rightarrow \tfrac{1}{2}(\lambda+1) f\Theta' + \underbrace{\tfrac{1}{2}(\lambda+1)\Theta f' + \tfrac{1}{2}(\lambda-1)\Theta f' - 2\lambda\Theta f'}_{} = \Theta''$$

I'd expect these to cancel well maybe not

$$\Rightarrow \Theta'' = \tfrac{1}{2}(\lambda+1) f\Theta' - \lambda\Theta f'$$

wouldn't bet on the algebra

Also $\psi_x = -Ra(T-T_\infty)$ so

$$Ra\, y^{\lambda} f' = -Ra\, y^{\lambda}\, \Theta$$

$$\Rightarrow \underline{f' = -\Theta}$$

(c) well $uT_x + vT_y = T_{xx}$

or $(uT)_x + (vT)_y = T_{xx}$

so $\cancel{\frac{d}{dx}}$ ~~int~~ or $[u(T-T_\infty)]_x + [v(T-T_\infty)]_y = T_{xx}$

$$u(T-T_\infty)\Big|_0^\infty + \frac{d}{dy}\int_0^\infty v(T-T_\infty)\,dx = T_x\Big|_0^\infty$$

$u=0$, $\cancel{T_x=}$ at $x=0$

$T=T_\infty$ at $x\to\infty$

$T_x=0$

$$\Rightarrow \underline{\frac{d}{dy}\int_0^\infty v(T-T_\infty)\,dx = -T_x\Big|_{x=0}}$$

~~Heater~~ heat added at wall is supplied to excess heat

within thermal plume

Comments.

1. Von Mises transforms from ~~y, x to ψ, x~~ x, y to ψ, y

thus $\dfrac{\partial}{\partial x} = \psi_x \dfrac{\partial}{\partial \psi}$

$$\frac{\partial}{\partial y} \to \frac{\partial}{\partial y} + \psi_y \frac{\partial}{\partial \psi}$$

& then $u \cdot -v T_\psi + v\left[T_y + u T_\psi\right] = T_{xx} = -v\dfrac{\partial}{\partial \psi}\left(-v\dfrac{\partial T}{\partial \psi}\right)$

$$\Rightarrow \quad T_y = \frac{\partial}{\partial \psi}\left[v \frac{\partial T}{\partial \psi}\right]$$

This can be handy if $v = v(y)$

well I suppose have $v = -\psi_x = Ra(T - T_\infty)$

so we do have $\dfrac{\partial T}{\partial y} = Ra \dfrac{\partial}{\partial \psi}\left[(T - T_\infty)\dfrac{\partial T}{\partial \psi}\right]$

And we might try $T - T_\infty = y^\lambda G(\xi)$, $\xi = \dfrac{\psi}{y^\beta}$

in which case $\lambda y^{\lambda-1} G - \beta \xi y^{\lambda-1} G' = Ra\, y^{2\lambda - 2\beta}\left[G G'\right]'$

so choose $\lambda - 1 = 2\lambda - 2\beta$

i.e. $\beta = \frac{1}{2}(\lambda + 1)$

& then $\lambda G - \frac{1}{2}(\lambda + 1)\xi G' = Ra\,(G G')'$

Comment

2. A check may be

$$\frac{d}{dy}\int_0^\infty v(T-T_\infty)\,dx = -T_x\big|_{x=0}$$ in similarity coordinates

$v = -\psi_x = -Ra\, y^{\lambda} f'$

$dx = \dfrac{d\eta}{Ra^{1/2}\, y^{\frac{1}{2}(\lambda-1)}}$

So $$\int_0^\infty v(T-T_\infty)\,dx = \int_0^\infty -Ra\, y^{\lambda} f' y^{\lambda}\Theta \frac{d\eta}{Ra^{1/2}\, y^{\frac{1}{2}(\lambda-1)}}$$

$$= -Ra^{1/2}\, y^{\frac{3}{2}\lambda+\frac{1}{2}}\int_0^\infty f'\Theta\, d\eta$$

So we want $$\left(\tfrac{3}{2}\lambda+\tfrac{1}{2}\right)Ra^{1/2}\, y^{\frac{3}{2}\lambda-\frac{1}{2}}\int_0^\infty \Theta f'\, d\eta = T_x\big|_{x=0} = y^{\lambda} Ra^{1/2}\, y^{\frac{1}{2}(\lambda-1)}\,\Theta'\big|_{\eta=0}$$

i.e. $$\Theta'\big|_{\eta=0} = \left(\tfrac{3}{2}\lambda+\tfrac{1}{2}\right)\int_0^\infty \Theta f'\, d\eta$$

note from $$\Theta'' = \tfrac{1}{2}(\lambda+1)\Theta' f - \lambda\Theta f'$$
$$= \tfrac{1}{2}(\lambda+1)(f\Theta)' - \tfrac{1}{2}(\lambda+1)\Theta f' - \lambda\Theta f'$$
$$= \tfrac{1}{2}(\lambda+1)(f\Theta)' - \left(\tfrac{3}{2}\lambda+\tfrac{1}{2}\right)\Theta f'$$

integrate 0 to ∞
$$-\Theta'\big|_{\eta=0} = -\left(\tfrac{3}{2}\lambda+\tfrac{1}{2}\right)\int_0^\infty \Theta f'\, d\eta$$

since $f=0$ on $\eta=0$ $\square$.