C5.7 (C6.4a) Topics in fluids 2014 q 3

C'

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$$p_z = -\rho$$

note $\varepsilon = Ro$

$\dot{u} = \frac{du}{dt}$ etc

$\frac{d}{dt} = \partial_t + u\partial_x + v\partial_y + w\partial_z$

$$\varepsilon \dot{u} - (1+\varepsilon\beta y) v + p_x = 0$$

$$\varepsilon \dot{v} + (1+\varepsilon\beta y) u + p_y = 0$$

$$u_x + v_y + w_z = 0$$

$$\varepsilon \dot{\rho} - N^2 w = 0$$

(a) $\varepsilon \ll 1$, $N, \beta \sim 1$ $\quad u = u_0 + \varepsilon u_1 \ldots$

(i) Leading order $\quad v = p_x \quad u = -p_y \quad$ geostrophy

$\Rightarrow u = \psi_y,\ v = -\psi_x$ if $\psi = -p$ $\quad$ (ρ w=0 so constant)

(ii) next order

$$p_{1z} = -\rho_1$$

$$\dot{u}_0 - \beta y v_0 - v_1 + p_{1x} = 0$$

$$\dot{v}_0 + \beta y u_0 + u_1 + p_{1y} = 0$$

$$u_{1x} + v_{1y} + w_{1z} = 0$$

$$\dot{\rho}_0 = N^2 w_1$$

& now $\dot{} \equiv \frac{d}{dt} = \partial_t + u_0\partial_x + v_0\partial_y$ $\quad$ (as $w_0 = 0$)

(b) curl of momentum:

$$\frac{\partial}{\partial y}\left[\frac{\partial u_0}{\partial t} + u_0 u_{0x} + v_0 u_{0y}\right] - \beta(y v_0)_y - v_{1y} + p_{1xy}$$

$$-\frac{\partial}{\partial x}\left[v_{0t} + u_0 v_{0x} + v_0 v_{0y}\right] - \beta(y u_0)_x - u_{1x} - p_{1xy}$$

$$= 0$$

Thus

$$u_{0yt} + u_{0y}u_{0x} + u_0 u_{0xy} + v_{0y}u_{0y} + v_0 u_{0yy} - \beta v_0 - \beta y v_{0y} - v_{1y}$$

$$- v_{0xt} - u_{0x}v_{0x} - u_0 v_{0xx} - v_{0x}v_{0y} - v_0 v_{0xy} \quad - \beta y u_{0x} - u_{1x}$$

$$= 0$$

$$\Rightarrow \frac{\partial}{\partial t}(D_H^2\psi) + u_0\underbrace{(u_{0y}-v_{0x})_x}_{D_H^2\psi} + \underbrace{(u_{0x}+v_{0y})}_{=0}(u_{0y}-v_{0x}) + v_0\underbrace{(u_{0y}-v_{0x})_y}_{D_H^2\psi}$$

$$D_H^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \qquad - \beta v_0 - \beta y \underbrace{(u_{0x}+v_{0y})}_{=0} - (u_{1x}+v_{1y}) = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x} + v_0\frac{\partial}{\partial y}\right)(\psi_{xx}+\psi_{yy}) - \beta v_0 + w_{1z} = 0$$

note $\frac{dy}{dt} = v_0$ $\quad \left(\frac{d}{dt} = \partial_t + u_0\partial_x + v_0\partial_y\right)$

& $w_1 = \frac{1}{N^2}\dot{p}_0 = -\frac{1}{N^2}\frac{d}{dt}(p_z) = \frac{1}{N^2}\frac{d}{dt}\psi_z = \frac{1}{N^2}\left(\partial_t + u_0\partial_x + v_0\partial_y\right)\psi_z$

$$\Rightarrow w_{1z} = \frac{1}{N^2}\frac{d}{dt}\psi_{zz}$$

So $\boxed{\frac{d}{dt}\left[\psi_{xx}+\psi_{yy} - \beta y + \frac{1}{N^2}\psi_{zz}\right] = 0}$

since $\frac{\partial}{\partial z}$ $u_{0z}\psi_{xz} + v_{0z}\psi_{yz}$
$= -u_{0z}v_{0z} + v_{0z}u_{0z} = 0$

(c) $\psi = y + \Psi$, linearise (base state $u=1$, $v=0$)

(i) $\frac{d}{dt} = \frac{\partial}{\partial t} + (1+\Psi_y)\frac{\partial}{\partial x} - \Psi_x\frac{\partial}{\partial y}$

$$\left[\partial_t + (1+\Psi_y)\partial_x - \Psi_x\partial_y\right]\left[\Psi_{xx}+\Psi_{yy} - \beta y + \frac{1}{N^2}\Psi_{zz}\right] = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left[\Psi_{xx}+\Psi_{yy}+\frac{1}{N^2}\Psi_{zz}\right] + \beta\Psi_x = 0$$

(ii)

$$\Psi = \exp[i(kx + ly + mz - \omega t)]$$

$$\Rightarrow (-i\omega + ik)\left[-k^2 - l^2 - \frac{m^2}{N^2}\right] + ik\beta = 0$$

$$-(\omega - k)\left[k^2 + l^2 + \frac{m^2}{N^2}\right] - ik\beta = 0$$

$$\omega = k - \frac{k\beta}{k^2 + l^2 + \frac{m^2}{N^2}}$$

Zonal wave speed is $\frac{\omega}{k} = 1 - \frac{\beta}{k^2 + l^2 + \frac{m^2}{N^2}}$

(iii) $\beta = 0$ $\omega = k$ wave speed is $\frac{\omega}{k}$ ~~kk~~ $= 1$

which is the zonal flow speed

ie waves move with the flow

(iv) $\beta > 0$ $\frac{\omega}{k} =$ ~~less~~ $1 - \frac{\beta}{k^2 + l^2 + \frac{m^2}{N^2}}$

so waves move ~~east~~ west wards relative to mean flow.