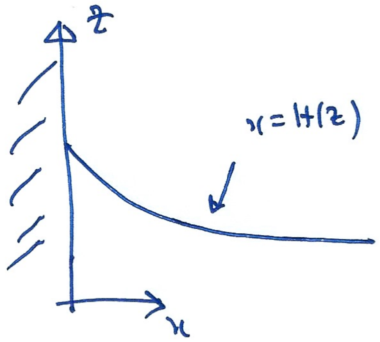
C8.7 Topics in fluids 2015 Q1

1.



$\rho g z = \sigma \kappa$

(a) Laplace-Young equation describes the meniscus of a fluid at a wall.

As shown, the hydrostatic pressure in a fluid is

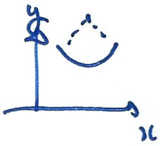
$p = p_0 - \rho g z$ and at the free surface

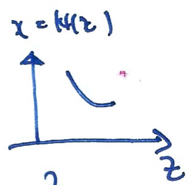
$p_a = p + \sigma \kappa$

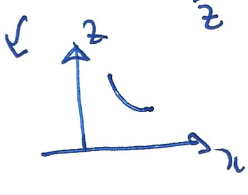
where σ is surface tension and κ is curvature, measured from the atmospheric side ($\kappa > 0$ as shown)

$\Rightarrow \quad p_a = p_0 - \rho g z + \sigma \kappa$

& taking $p_0 = p_a$, $\Rightarrow$ $\rho g z = \sigma \kappa$

Normally  as defined $\kappa = \dfrac{y''}{(1+y'^2)^{3/2}}$

or  $\kappa = \dfrac{H''}{(1+H'^2)^{3/2}}$ with $H(z)$



So we can write L-Y as

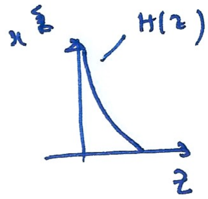
$$\rho g z = \frac{\sigma H''}{(1+H'^2)^{3/2}}$$

we scale $z \sim H \sim l$

$$\Rightarrow \text{ non-d } \quad \rho g l z = \frac{\sigma}{l} \frac{H''}{(1+H'^2)^{3/2}}$$

so choose $l = \left(\frac{\sigma}{\rho g}\right)^{1/2}$ (i.e. Bond number $= 1$)

$z \to 0$ at ∞



$$\Rightarrow \quad z = \frac{H''}{(1+H'^2)^{3/2}}$$

integrate $\quad \frac{1}{2} z^2 + A = \frac{H'}{(1+H'^2)^{1/2}}$

we are told

$$\left[\left\{\frac{H'}{(1+H'^2)^{1/2}}\right\}' = \frac{H''}{(1+H'^2)^{1/2}} - \frac{H'^2 H''}{(1+H'^2)^{3/2}} = \frac{H''}{(1+H'^2)^{3/2}}\right]$$

$H' \to -\infty$ as $z \to 0$

$$\Rightarrow \frac{H'}{(1+H'^2)^{1/2}} \to -1 \quad \left(\simeq \frac{H'}{|H'|}\right)$$

So $A = -1$

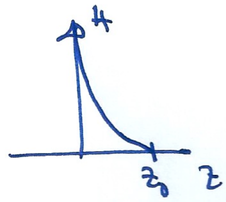
So as $z \to 0$ $\quad \frac{1}{2} z^2 - 1 = \frac{-1}{\left(1+\frac{1}{H'^2}\right)^{1/2}}$ $\quad$ (– sign as dividing by $|H'|$)

$$1 - \frac{1}{2} z^2 = \left(1 + \frac{1}{H'^2}\right)^{-1/2} \simeq 1 - \frac{1}{2H'^2} \ldots$$

$\Rightarrow H'^2 \simeq \frac{1}{z^2} \quad \Rightarrow H' = -\frac{1}{z} \quad H \sim -\ln z$ $\quad$ oops not asked that

$$\frac{H'}{(1+H'^2)^{1/2}} = -(1-\tfrac{1}{2}z^2)$$

So we need $z<\sqrt{2}$ for $H'<0$



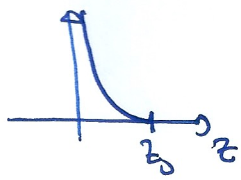
$$H'^2 = (1+H'^2)(1-\tfrac{1}{2}z^2)^2$$

$$H'^2\left[1-(1-\tfrac{1}{2}z^2)^2\right] = (1-\tfrac{1}{2}z^2)^2$$

$$H' = \frac{-(1-\frac{1}{2}z^2)}{\left[1-(1-\frac{1}{2}z^2)^2\right]^{1/2}}$$

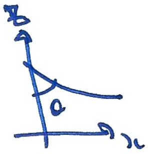
$$\Rightarrow \quad H = \int_z^{z_0} \frac{(1-\frac{1}{2}\zeta^2)\,d\zeta}{\left[1-(1-\frac{1}{2}\zeta^2)^2\right]^{1/2}}$$

He wants $H \sim \dfrac{(z-z_0)^2}{\sqrt{2}}$ as $z \to z_0$??



– think this is just wrong.

[normally you prescribe a contact angle θ :



& thus $H' = -\tan\theta$ at $z=z_0$

If we define $1-\frac{1}{2}z_0^2 = \sin\phi$

then $H'(z_0) = -\tan\phi$ so $\phi=\theta$

& $1-\frac{1}{2}z_0^2 = \sin\theta \Rightarrow z_0$.].

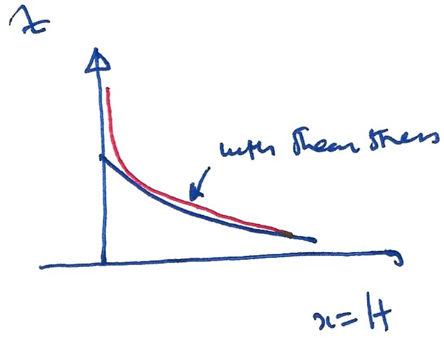
If you say that $\theta=0$ so contact is vertical then

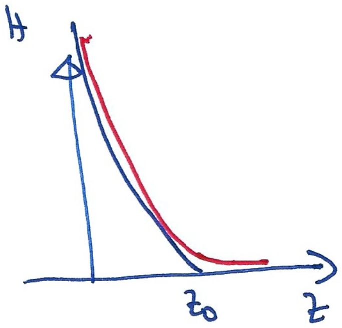
$z_0=\sqrt{2}$ and $H' \approx -(1-\frac{1}{2}z^2)$ near $z=z_0=\sqrt{2}$

so $H' \approx -\frac{1}{2}(\sqrt{2}+z)(\sqrt{2}-z) \approx -\frac{1}{2}\cdot 2\sqrt{2}(\sqrt{2}-z) \approx \sqrt{2}(z-\sqrt{2})$

so $H \approx \frac{\sqrt{2}}{2}(z-\sqrt{2})^2 = \dfrac{(z-\sqrt{2})^2}{\sqrt{2}}$ as required. He needed to say $\theta=0$.

(b)





now $\left[\frac{H^3}{3}(H_{zzz}-1)+\delta H^2\right]_z = 0$ $\delta \ll 1$

outer approx $z<z_0$ is $H = \int_z^{z_0} \frac{(1-\frac{1}{2}\zeta^2)\,d\zeta}{[1-(1-\frac{1}{2}\zeta^2)^2]^{1/2}}$

[well posed(?) was not thin film so strictly this is incorrect]

Assume $H \sim \frac{(z-z_0)^2}{\sqrt{2}}$ $(\& \ z_0=\sqrt{2})$

as $z \to z_0$

$z = z_0 + \epsilon Z$ $H = \eta h$

$\Rightarrow$ $\left[\frac{\eta^3 h^3}{3}\left(\frac{\eta}{\epsilon^3} h_{ZZZ} - 1\right) + \delta \eta^2 h^2\right]_Z = 0$

~~choose $\eta^4 = \epsilon^3$, $\epsilon = \delta\eta^2$~~

$\Rightarrow$ $\cancel{\left[\frac{h^3}{3}(\epsilon h_{ZZZ} - 1) + \right.}$

$\Rightarrow$ $\left[\frac{h^3}{3}\left(\frac{\eta^4}{\epsilon^3} h_{ZZZ} - \eta^3\right) + \delta\eta^2 h^2\right]_Z = 0$

choose $\frac{\eta^4}{\epsilon^3 \delta \eta^2} = 1$ $\Rightarrow$ $\eta^2 = \epsilon^3 \delta$

Note to match to $H \sim \frac{(z-z_0)^2}{\sqrt{2}}$

we need $\eta h \sim \frac{\varepsilon^2 Z^2}{\sqrt{2}}$

so $\eta = \varepsilon^2$

$\Rightarrow \varepsilon = \delta$ & $\eta = \delta^2$

$$\Rightarrow \left[\frac{h^3}{3}\left(\frac{\delta^8}{\delta^3} h_{ZZZ} - \delta^6\right) + \delta^5 h^2\right]_Z = 0$$

ie $\left[\frac{h^3}{3}\left(h_{ZZZ} - \delta\right) + h^2\right]_Z = 0$

with $h \sim \frac{Z^2}{\sqrt{2}}$ as $Z \to -\infty$

Lead order $\frac{h^3}{3} h_{ZZZ} + h^2 \sim$ constant $= p^2$ say

if $h \to p$ as $Z \to \infty$

(then $H = \eta h = \delta^2 h \to \delta^2 p$ as $Z \to \infty$)

have to solve $\frac{h^3}{3} h''' + h^2 = p^2$, $h \sim \frac{Z^2}{\sqrt{2}}$ as $Z \to -\infty$

$h \to p$ as $Z \to +\infty$

Let $f(\zeta)$ satisfy $\frac{f^3}{3} f_{\zeta\zeta\zeta} + f^2 = 1$ $\quad$ $f \to 1$ as ∞

$f \sim p_0 \zeta^2$ as $\zeta \to -\infty$

Seek a solution

$$h(Z) = a f(bZ) \quad , \quad \zeta = bZ$$

$$\Rightarrow \quad a^3 \frac{f^3}{3} a b^3 f''' + a^2 f^2 = p^2$$

$$\Rightarrow \quad a^4 b^3 = a^2 = p^2 \qquad \Rightarrow a = p, \quad b = \frac{1}{p^{2/3}}$$

and

$$h \sim a p_0 b^2 Z^2 \quad \text{as } Z \to -\infty$$

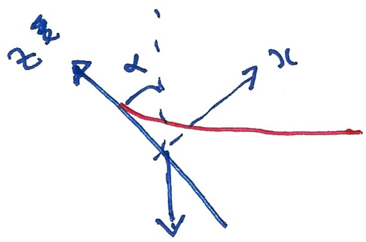
$$\Rightarrow a b^2 p_0 = \frac{1}{\sqrt{2}}$$

$$\text{so } b = a^{-2/3} \quad , \quad a^{-1/3} p_0 = \frac{1}{\sqrt{2}}$$

$$\Rightarrow a = (\sqrt{2} p_0)^3$$

$$\Rightarrow p = (\sqrt{2} p_0)^3$$

(c)



$$\underline{g} = g(-\sin\alpha, -\cos\alpha)$$

The new coordinates

~~$(\tilde{x}, \tilde{z})$ (X, Z) say $= (x, z) e^{i\alpha}$~~

~~So $(\tilde{H}, \tilde{z}) = (H, z) e^{i\alpha}$~~

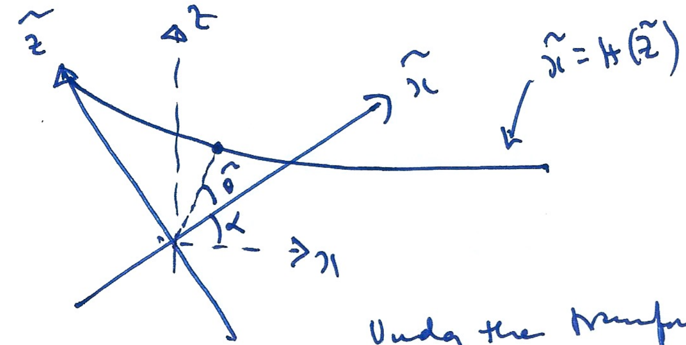
~~HAA~~. $\tilde{x} + i\tilde{z} = (x + iz) e^{i\alpha}$

~~$\Rightarrow \quad H + iz = (\tilde{H} + i\tilde{z}) e^{-i\alpha}$~~

~~to the new coordinates~~

~~$\Rightarrow \quad H = \tilde{H}\cos\alpha + \tilde{z}\sin\alpha, \quad z = \tilde{z}$~~

(c)



$\tilde{\theta} = \theta - \alpha$ ~~$re^{i\tilde{\theta}}$~~ $= re^{i(\tilde{\theta}+\alpha)}$

Under the transformation
$re^{i\tilde{\theta}} = \tilde{x} + i\tilde{z} = (x+iz)\, e^{-i\alpha}$

or $x + iz = (\tilde{x} + i\tilde{z})\, e^{+i\alpha}$

we have $z = \tilde{z}\cos\alpha + \tilde{x}\sin\alpha$

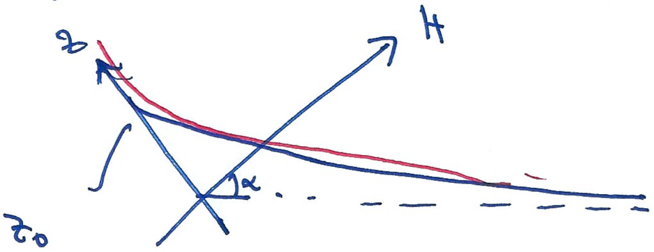
The dimensionless eq L-Y eq is $z = \kappa$ ← curvature

& κ has the same form (property of curve not of coordinates)

=> in new coordinates

$$z\cos\alpha + H\sin\alpha = \frac{H''}{(1+H'^2)^{3/2}}$$

~~be the ... H ...~~



$z \sim -H\tan\alpha$

i.e. $z\cos\alpha + H\sin\alpha \to 0$ as $z \to -\infty$ }

~~$H z_0$~~ $H = 0$ at z_0, $z_0\cos\alpha = (2 - 2\sin\alpha)^{1/2}$

(note $\alpha = 0 \Rightarrow z_0 = \sqrt{2}$ as earlier)

ah think f... again

This is a bit vague. Actually very vague

The connection of $z = \frac{H_{zz}}{(1+H_z^2)^{3/2}}$ and the thin film theory

must be

$$\frac{\partial}{\partial z}\left[\frac{H^3}{3}\frac{\partial}{\partial z}\left\{\frac{H_{zz}}{(1+H_z^2)^{1/2}} - z\right\} + \delta H^2\right] = 0$$

$H_z \ll 1$

$$\frac{\partial}{\partial z}\left[\frac{H^3}{3}\left\{H_{zzz} - 1\right\} + \delta H^2\right] = 0$$

so would have replace z by $z\cos\alpha + H\sin\alpha$

thus $\frac{\partial}{\partial z}\left[\frac{H^3}{3}\left\{H_{zzz} - (\cos\alpha + H_z\sin\alpha)\right\} + \delta H^2\right]$

but rescaled we lose anyway

I think we want the outer solution

If we assume as before $H \sim c(z-z_0)^2$ as $z \to z_0$

then $H_z \to 0$, $H_{zz} \sim 2c$

so $2c = z_0\cos\alpha$

$$\Rightarrow H \sim \cancel{\frac{(z-z_0)^2}{2}}\ \tfrac{1}{2} z_0 \cos\alpha\,(z-z_0)^2$$

rescaling in precursor film as before same eq

$$\frac{h^3}{3}h''' + h^2 = R^2 \quad \text{where } h \to R \text{ at } \infty$$

but now

$H = \delta^2 h$, ~~tt~~ $h \sim \frac{1}{2} z_0 \cos\alpha \; Z^2 \quad Z \to -\infty$ (9)

So solve $\quad \frac{h^3}{3} h''' + h^2 = R^2 \quad , \quad h \sim \frac{1}{2} z_0 \cos\alpha \; Z^2 \quad Z \to -\infty$

try $h = a f(bZ) \quad \zeta = bZ$

$\Rightarrow \quad a^4 b^3 = a^2 = R^2$

& $h \sim a p_0 b^2 Z^2, \quad Z \to -\infty$

$\Rightarrow \quad b = a^{-2/3}$

$a^{-1/3} p_0 = \frac{1}{2} z_0 \cos\alpha$

$$a = \left(\frac{2 p_0}{z_0 \cos\alpha} \right)^3$$

& so $R = \left(\frac{2 p_0}{z_0 \cos\alpha} \right)^3 = \left(\frac{p_0 \sqrt{2}}{\sqrt{1 - \sin\alpha}} \right)^3$

I suppose for $0 < \alpha < \pi/2$.

[?]