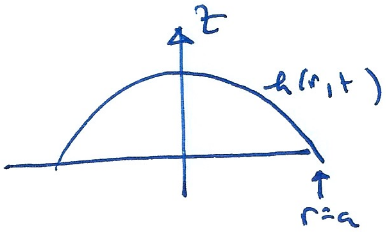
CS.7 2015 q 2



$$2\pi \int \phi h r\,dr = V_0$$

(a)

$$\phi h_t = \frac{k\rho g}{\mu} \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial h^2}{\partial r}\right)$$

$$h, r \sim l = \left(\frac{V_0}{2\pi\phi}\right)^{1/3}$$

$$t \sim \left(\frac{\phi^2 V_0}{2\pi}\right)^{1/3} \frac{\mu}{k\rho g} = \frac{\mu\phi}{k\rho g}\, l$$

$$\Rightarrow \phi l \cdot \frac{k\rho g}{\mu\phi l} h_t = \frac{k\rho g}{\mu} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h^2}{\partial r}\right) \qquad \text{non-d}$$

ie $$h_t = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial h^2}{\partial r}\right)$$

Also $$2\pi \int \phi l^3\, r h\,dr = V_0 \qquad \text{non-d}$$

$$= \int_0^a r h\,dr = 1 \qquad (\text{so } I = 1)$$

$$\frac{d}{dt}\int_0^a r h\,dr = \int_0^a r h_t\,dr = \left[r\frac{\partial h^2}{\partial r}\right]_0^a = 0$$

(if $hh_r = 0$ when $h=0$)

(2

(b) $h = t^\alpha H(\eta)$ $\eta = \frac{r}{t^\beta}$, $a = t^\beta \zeta$

$\Rightarrow$ $\cancel{t^\alpha H - \frac{\beta\eta}{t^{\alpha+1}} \frac{H'}{t^\alpha}}$

$+\alpha t^{\alpha-1} H - \beta\eta t^{\alpha-1} H' = t^{2\alpha-2\beta} \frac{1}{\eta}\frac{d}{d\eta}\left[\eta \frac{d H^2}{d\eta}\right]$

So $\alpha - 1 = 2\alpha - 2\beta$

AND $\int_0^a r h\, dr = 1 = \int_0^\zeta t^{2\beta+\alpha} \eta H \, d\eta$

So $\alpha + 2\beta = 0$

$\Rightarrow \alpha = -2\beta$, $\underline{\alpha = -\frac{1}{2}, \ \beta = \frac{1}{4}}$

$\Rightarrow$ $\cancel{t\alpha - \beta\eta}$ $+ \alpha H - \beta\eta H' = \frac{1}{\eta}\left[\eta (H^2)'\right]'$

$\Rightarrow -\frac{1}{2}\eta H - \frac{1}{4}\eta^2 H' = \left[\eta (H^2)'\right]'$

$= -\left(\frac{1}{4}\eta^2 H\right)'$

So $\eta (H^2)' = -\frac{1}{4}\eta^2 H$ to satisfy $H'(0) = 0$

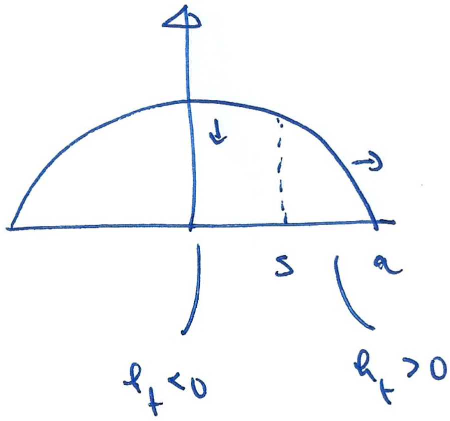
$\Rightarrow 2\eta H H' = -\frac{1}{4}\eta^2 H$

$H' = -\frac{1}{8}\eta$

So $\underline{H = \frac{1}{16}(\zeta^2 - \eta^2)}$

$\int_0^\zeta \eta H \, d\eta = 1 = \frac{1}{16}\left[\zeta^2\eta - \frac{1}{3}\eta^3\right]_0^\zeta = 1 = \frac{\zeta^3}{24}$ $\Rightarrow \underline{\zeta = 24^{1/3}}$

(c)



So $h_t = \frac{1+\varepsilon}{r}(r h_r^2)_r \quad 0 < r < s$

$h_t = \frac{1}{r}(r h_r^2)_r \quad s < r < a$

$$I = \int_0^a r h \, dr = \int_0^s + \int_s^a r h \, dr$$

$$\dot{I} = (1+\varepsilon)\, r h_r^2 \Big|_0^s + r h_r^2 \Big|_s^a$$

$$= s\left[2(1+\varepsilon) h h_r - 2 h h_r\right]\Big|_s$$

$$= \underline{2\varepsilon s h h_r \Big|_s}$$

As before (in eq.) $\alpha + 1 = 2\beta$, $h = t^\alpha H(\eta)$

gives $\alpha H - \beta \eta H' = \frac{1}{\eta}\left[\eta (H^2)'\right]'$ or $\frac{1+\varepsilon}{\eta}\left[\eta (H^2)'\right]'$

Then $I = \int_0^\zeta t^{2\alpha+1} \eta H \, d\eta \qquad a = t^{\frac{\alpha+1}{2}} \zeta \quad \left(\beta = \frac{\alpha+1}{2}\right)$

$s = t^{\frac{\alpha+1}{2}} \sigma$

And with ζ constant

$$\dot{I} = (2\alpha+1)\, t^{2\alpha} \int_0^\zeta \eta H \, d\eta$$

$$= 2\varepsilon s h h_r \Big|_s$$

Thus

$$\dot{I} = (2\alpha+1)\, t^{2\alpha} \int_0^{\zeta} \eta H \, d\eta$$

$$= 2\varepsilon t^{\beta} \sigma \, t^{\alpha} H \, t^{\alpha-\beta} H' \quad \text{at } \eta = \sigma$$

$$= 2\varepsilon t^{2\alpha} \sigma H H'$$

so $$(2\alpha+1)\int_0^{\zeta} \eta H \, d\eta = 2\varepsilon\sigma(HH')\big|_{\eta=\sigma}$$

(d)

$\varepsilon \to 0$ At leading order $I = 1$ gives first solution

We have to solve

$$\alpha H - \beta \eta H' = \frac{(1+\varepsilon)}{\eta}\left[\eta (H^2)'\right]' \qquad 0 < \eta < \sigma, \quad H'(0) = 0$$

$$\alpha H - \beta \eta H' = \frac{1}{\eta}\left[\eta (H^2)'\right]' \qquad H(\zeta) = 0 \quad ; \; H, H' \text{ cts at } \eta = \sigma$$

note $h_t = \alpha t^{\alpha-1} H - \beta \eta t^{\alpha-1} H' = 0$ at $\eta = \sigma$

so $\alpha H - \beta \sigma H' = 0$ at $\eta = \sigma$

Expanding etc $\alpha = \alpha_0 + \varepsilon \alpha_1 \dots \quad H = H_0 + \varepsilon H_1 + \dots$

Lead order $H_0 = \frac{1}{16}(\zeta_0^2 - \eta^2) \qquad \zeta_0 = 24^{1/3}, \; \alpha_0 = -\frac{1}{2} \quad \beta_0 = \frac{1}{2}(\alpha_0 + 1) = +\frac{1}{4}$

At leading order

$$\alpha H - \beta \sigma H' = 0$$

$$\Rightarrow \; -\tfrac{1}{2} H_0 \cancel{} - \tfrac{1}{4} \sigma H_0' = 0 \qquad \text{at } \eta = \sigma$$

$$\Rightarrow \; -\tfrac{1}{2} \tfrac{1}{16} (\xi_0^2 - \sigma^2) - \tfrac{1}{4} \sigma \,.\, -\tfrac{1}{8} \sigma \; = 0$$

$$\Rightarrow \; -\xi_0^2 + \sigma^2 + \sigma^2 = 0$$

$$\sigma_0 = \frac{\xi_0}{\sqrt{2}}$$

Now $\quad (2\alpha+1) \int_0^{\xi} \eta H \, d\eta \; = \; 2\varepsilon \sigma (HH')\Big|_{\eta=\sigma}$

$2\alpha + 1 = 2\varepsilon \alpha_1 + \ldots$

So at leading order $\quad \int_0^{\xi_0} \eta H_0 \, d\eta = 1 \;, \; H_0 = \tfrac{1}{16}(\xi_0^2 - \eta^2) \quad H_0' = -\tfrac{1}{8}\eta$

$$2\alpha_1 \; = \; 2\sigma_0 \,.\, \tfrac{1}{16} (\xi_0^2 - \sigma_0^2) \,.\, -\tfrac{1}{8}\sigma_0$$

$$\Rightarrow \alpha_1 \; = \; -\frac{1}{128} \; \frac{\xi_0}{\sqrt{2}} \,.\, \tfrac{1}{2}\xi_0^2 \qquad \qquad \& \; \xi_0 = 24^{1/3}$$

$$= \; -\frac{24}{128 \,.\, 2\sqrt{2}} \; = \; -\frac{3}{32\sqrt{2}}$$

The front is at $\quad a = \xi t^{\frac{1}{2}(\alpha+1)} \; = \xi t^{\frac{1}{4} - \frac{3\varepsilon}{32\sqrt{2}} \ldots}$

so it slows down