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2015 q3 CS.7 Topics in fluids

3 (a) This is the same as 2017 q 3(a) !!

(b)

$$\alpha\rho_g(v_t+vv_x) = -\alpha p_x - \alpha\rho_g g - \frac{M}{A}$$

$$\rho_l\left[\{(1-\alpha)u\}_t + D_l\{(1-\alpha)u^2\}_x\right] = -(1-\alpha)p_x - (1-\alpha)\rho_l g + \frac{M}{A}$$

bubbly flow:

drag on single bubble is $C_D\pi a^2\rho_l(v-u)|v-u| = D$ say

& we take

$$\frac{M}{A} = \frac{3C_D\rho_l\alpha}{4a}|v-u|(v-u)$$

$$= \frac{3\alpha}{4a}\cdot C_D\rho_l|v-u|(v-u)$$

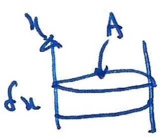
$$= \frac{3\alpha}{4a}\,\frac{D}{\pi a^2} = \frac{\alpha D}{\frac{4}{3}\pi a^3}$$

If α is the volume fraction & the bubbles have radius a,

then $\frac{\alpha}{\frac{4}{3}\pi a^3}$ is the number of bubbles / unit volume (= n say)

so nD is drag force on bubbles per unit volume, $= \frac{M}{A}$

In deriving momentum, we have (in a tube)



$$\frac{d}{dt}\int\Big[\alpha\underbrace{\rho_g}_{\text{density}}v\underbrace{A\,\delta x}_{\text{volume}}\Big] = \text{rate of change of momentum} \ldots$$

$$= \ldots -\frac{M}{A}\cdot A\,\delta x$$

whence $\frac{\partial}{\partial t}(\alpha\rho_g v)\ldots = -\frac{M}{A}$ as given above.

(As the mass of the bubbles is conserved,

$\frac{4\pi}{3}\rho_g a^3$ = constant, $a \propto \rho_g^{-1/3}$.)

<u>Scale</u> $x \sim L$, $p - p_0 \sim \rho_\ell g L$, $u, v \sim U = \sqrt{\frac{4a_0 g}{3c_D}}$

$\rho_g \sim \rho_0$, $t \sim \frac{L}{U}$

$$\Rightarrow \quad \alpha \rho_0 \rho_g \frac{U^2}{L}(v_t + v v_x) = -\alpha \rho_\ell g p_x - \alpha \rho_0 \rho_g g - \frac{M}{A}$$

Note $\frac{M}{A} = \frac{3c_D \rho_\ell \alpha}{4a}(v-u)|v-u|$

non-d

$$\text{non-d} = \frac{3c_D \rho_\ell \alpha U^2}{4a_0} \underline{|v-u|(v-u)}\, \rho_g^{1/3}$$

$$\& \quad U^2 = \frac{4a_0 g}{3c_D} \quad \Rightarrow \quad \frac{3c_D U^2}{4a_0} = g$$

So $\frac{M}{A} = \alpha \rho_\ell g \rho_g^{1/3} |v-u|(v-u)$ think I'll write $\rho_g = \rho$ for a bit

non-d non-d

$$\Rightarrow \text{gas momentum is} \quad \frac{\rho_0 U^2}{\rho_\ell g L}\rho(v_t + v v_x) = -p_x - \frac{\rho_0}{\rho_\ell}\rho - \rho^{1/3}|v-u|(v-u)$$

Liquid momentum is

$$\rho_\ell \frac{U^2}{L}\left[\{(1-\alpha)u\}_t + D_\ell\{(1-\alpha)u^2\}_x\right] = -(1-\alpha)\rho_\ell g p_x - (1-\alpha)\rho_\ell g + \alpha \rho_\ell g \rho^{1/3}|v-u|(v-u)$$

$$\Rightarrow \quad \frac{U^2}{gL}\left[\{(1-\alpha)u\}_t + D_\ell\{(1-\alpha)u^2\}_x\right] = -(1-\alpha)p_x - (1-\alpha) + \alpha \rho^{1/3}|v-u|(v-u)$$

If $\rho_0 \ll \rho_\ell$ ... what of $\frac{U^2}{gL}$?

Note $\frac{U^2}{gL} = \frac{4a_0}{3c_D L} \sim \frac{a_0}{L} \ll 1$

(we were told $c_D \sim O(1)$ & $a_0 \ll L$ — nice!)

So $\frac{U^2}{gL} \ll 1$, $\frac{\rho_0}{\rho_\ell} \ll 1$.

Losing the acceleration terms, and also the buoyancy term $\frac{\rho_g}{\rho_\ell}$ in gas momentum, the dimensional momentum eq^s are

~~ρ_x~~

$$0 = -\alpha p_x - \frac{M}{A}$$

$$0 = -(1-\alpha)p_x - (1-\alpha)\rho_\ell g + \frac{M}{A}$$

Add $p_x = -(1-\alpha)\rho_\ell g \underset{\substack{\uparrow \\ \text{(gas momentum)}}}{=} -\frac{M}{A\alpha} = -\frac{3c_D \rho_\ell |v-u|(v-u)}{4a}$ (2)

D

(c) Steady flow, $u=0$

mass is dimensional (gas) $\left[\frac{\partial}{\partial t}(\alpha\rho_g)\right] + \frac{\partial}{\partial x}(\alpha\rho_g v) = 0$

$\Rightarrow \alpha\rho_g v = \text{constant} = Q$ say

Then (2) $\Rightarrow (1-\alpha)g = \frac{3c_D v^2}{4a}$

and thus

$$\alpha^2 \rho_g^2 (1-\alpha) g = \frac{3c_D}{4a}(\alpha \rho_g v)^2 = \frac{3c_D Q}{4a_0}\left(\frac{\rho_g}{\rho_0}\right)^{1/3}$$

via definition of a

so
$$\alpha^2(1-\alpha) = \left[\frac{3c_D Q}{4a_0 \rho_g^2 g}\right]\left(\frac{\rho_g}{\rho_0}\right)^{1/3}$$
$$= \left(\frac{3c_D Q}{4a_0 \rho_0^2 g}\right)\frac{\rho_0^2}{\rho_g^2}\left(\frac{\rho_g}{\rho_0}\right)^{1/3}$$
$$= \left(\frac{3c_D Q}{4a_0 \rho_0^2 g}\right)\left(\frac{\rho_g}{\rho_0}\right)^{-5/3}$$

$$\Rightarrow \frac{\rho_g}{\rho_0} = \left(\frac{3c_D Q}{4a_0 \rho_0^2 g}\right)^{3/5}\frac{1}{[\alpha^2(1-\alpha)]^{3/5}} \stackrel{\Delta}{\equiv} \left[\frac{\alpha_0^2(1-\alpha_0)}{\alpha^2(1-\alpha)}\right]^{3/5}$$

not very satisfactory as two choices of α_0 if $\frac{3c_D Q}{4a_0\rho_0^2 g} < \left(\frac{4}{27}\right)^{5/3}$ o/w $\alpha_0 < 0$!

If $p = c^2 \rho_g$

then $c^2 \frac{d\rho_g}{dx} = -\rho_l g(1-\alpha)$;

with $\rho_g = \rho_0 \rho$, $B = \left(\frac{3c_D Q}{4a_0\rho_0^2 g}\right)^{3/5}$, $\rho' = \frac{d\rho}{dx} = B \cdot \frac{-3/5}{[\alpha^2(1-\alpha)]^{8/5}}(2\alpha - 3\alpha^2)\,\alpha'$

$$\Rightarrow \frac{-\frac{3}{5}B(2\alpha-3\alpha^2)\alpha'}{[\alpha^2(1-\alpha)]^{8/5}} = -\frac{\rho_l g}{\rho_0 c^2}(1-\alpha) \quad \Rightarrow \frac{d\alpha}{dx} = \frac{5\rho_l g}{3\rho_0 c^2 B}\,\frac{\alpha^{11/5}(1-\alpha)^{13/5}}{\cancel{\alpha}(2-3\alpha)}$$