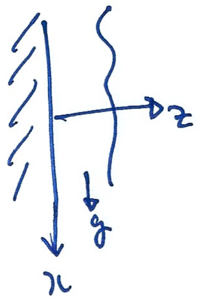
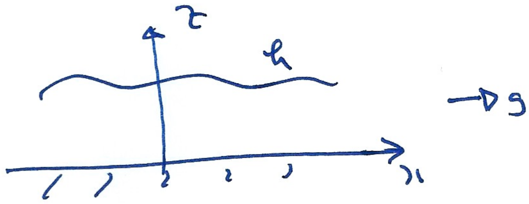
C5.7 Topics in fluids 2016 Q1

1. (a)



wlz



$p_x - 1 = u_{zz}$, $p_z = 0$

$z = h$: $p = -h_{xx}$

i $p = -h_{xx}$ ~~everywhere~~ everywhere

$u_{zz} = -1 - h_{xxx}$

$u_z = (1 + h_{xxx})(h - z)$

$u = (1 + h_{xxx})(hz - \frac{1}{2}z^2)$

ii $h_t + \frac{\partial}{\partial x}\int_0^h u\,dz = 0$

$\int_0^h u\,dz = (1 + h_{xxx})\cdot\frac{1}{3}h^3$

$\Rightarrow h_t = -\frac{\partial}{\partial x}\left[\frac{1}{3}h^3(1 + h_{xxx})\right]$

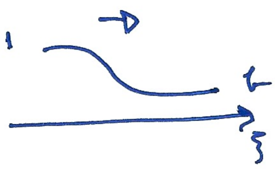
(b)

$h = H(\xi) \qquad \xi = x - ct$

$$\Rightarrow \quad cH' = \left[\tfrac{1}{3} H^3 (1 + H''') \right]'$$

$H \to 1 \quad \xi \to -\infty$

$H \to b \quad \xi \to +\infty$



$$\Rightarrow \quad c(H-1) = \tfrac{1}{3} H^3 (1 + H''') - \tfrac{1}{3} \qquad \text{for } H \to 1 \text{ at } -\infty$$

$$\Rightarrow \quad \frac{3c(H-1)}{H^3} = 1 + H''' - \frac{1}{H^3}$$

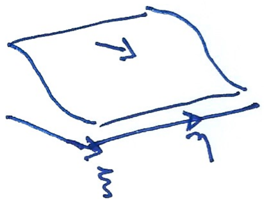
$$\Rightarrow \quad H''' = \frac{3c(H-1) + 1}{H^3} - 1$$

$H \to b$ at $+\infty$ $\Rightarrow$ $$0 = \frac{3c(b-1) + 1}{b^3} - 1$$

$$\Rightarrow \quad 3c(b-1) + 1 = b^3$$

$$c = \frac{1 - b^3}{3(1-b)} = \tfrac{1}{3}(1 + b + b^2)$$

(c)



$$h_t - c h_\xi = -h^2 h_\xi - \nabla \cdot \left[\frac{h^3}{3} \nabla \nabla^2 h \right]$$

$h \to 1 \quad \xi \to -\infty$

$h \to b \quad \xi \to \infty$

(i) $h = H + G$ linearise, $H = H(\xi)$

$$\Rightarrow \quad G_t - c\,G_\xi = -\frac{\partial}{\partial \xi}(H^2 G) - \underline{D}\cdot\left[H^2 G H''' \underline{i} + \tfrac{1}{3}H^3 \underline{D} D^2 G\right]$$

$$\left[\,\alpha\ \underline{D}\cdot\left[\tfrac{1}{3}(H+G)^3 \underline{D} D^2 (H+G)\right]\right.$$

$$\cancel{\underline{D}\cdot\left[H^2 G \underline{D} D^2 (H+G)\right)}$$

$$\left.\simeq \underline{D}\cdot\left[\left(\tfrac{1}{3}H^3 + H^2 G ..\right)\underline{D} D^2 (H+G)\right]\right]$$

$$\cancel{\underline{D}\cdot\left[\tfrac{1}{3}H^3 \underline{D} D^2 G\right.}$$

Put $G = g(\xi)\,e^{\lambda t + iky}$, $D^2 G = (g'' - k^2 g)\,e^{\lambda t + iky}$,

$$\Rightarrow \quad \lambda g - c g' = -(H^2 g)' - \left[H^2 H''' g\right]'$$

$$- \left\{\left[\tfrac{1}{3}H^3 (g''' - k^2 g')\right]' - \tfrac{1}{3}H^3 k^2 (g'' - k^2 g)\right\}$$

$g(\pm\infty) = 0$

long wave $k \ll 1$, $\lambda = \lambda_1 k^2 + ..$ $\quad g = \underbrace{H'}_{g_0} + g_1 k^2$

↳ will work as when $k = 0$

$g = \dfrac{H(\xi + \delta) - H(\xi)}{\delta}$

satisfies eq

$$\Rightarrow \lambda_1 k^2 \left[H' + g_1 k^2 ..\right] - c\left[H'' + g_1' k^2 ..\right]$$

$$= -\left[H^2 (H' + g_1 k^2 ..)\right]' - \left[H^2 H''' (H' + g_1 k^2 ..)\right]'$$

$$- \left[\tfrac{1}{3}H^3 \left(H'''' + g_1''' k^2 - k^2 H'' ...\right)\right]'$$

$$+ \tfrac{1}{3}H^3 k^2 \left[H''' + ..\right]$$

At $O(1)$

$$-cH'' = -(H^2H')' - \left[H^2 H''' H'\right]' - \left[\tfrac{1}{3}H^3 H''''\right]'$$

$$\Rightarrow -cH' = -H^2H' - \underbrace{H^2 H''' H' - \tfrac{1}{3}H^3 H''''}_{-\left(\frac{1}{3}H^3 H'''\right)'} \qquad (H' \to 0 \text{ at } \pm\infty)$$

so $\quad -cH = -\tfrac{1}{3}H^3 - \tfrac{1}{3}H^3 H''' \quad$ + constant as found earlier (p2)

At $O(k^2)$

$$\lambda_1 H' - cg_1' = -(H^2 g_1)' - (H^2 H''' g_1)' - \left[\tfrac{1}{3}H^3(g_1''' - H'')\right]' + \tfrac{1}{3}H^3 H'''$$

integrate from $-\infty$ to ∞

$$\lambda_1 (b-1) = +\tfrac{1}{3}\int_{-\infty}^{\infty} H^3 H''' \, d\xi$$

$$= (p2) \quad +\tfrac{1}{3}\int_{-\infty}^{\infty} H^3 \left\{ \frac{3c(H-1)+1}{H^3} - 1 \right\} d\xi$$

so $$\lambda_1 = -\frac{1}{3(1-b)} \int_{-\infty}^{\infty} \left[3c(H-1) + 1 - H^3\right] d\xi$$

$(c = \tfrac{1}{3}(1+b+b^2))$

$$= -\frac{1}{3(1-b)} \int_{-\infty}^{\infty} \left[(1+b+b^2)(H-1) + 1 - H^3\right] d\xi$$

$$\Rightarrow \lambda_1 = -\frac{1}{3(1-b)} \int_{-\infty}^{\infty} (H-1)\left\{1+b+b^2 - (1+H+H^2)\right\} d\xi$$

$(1-H^3) = (1-H)(1+H+H^2)$

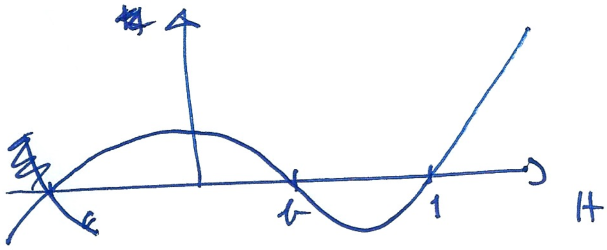
$$= -\frac{1}{3(1-b)} \int_{-\infty}^{\infty} (H-1)\left\{b-H+b^2-H^2\right\} d\xi$$

$$= \frac{1}{3(1-b)} \int_{-\infty}^{\infty} (H-1)\left[H^2-b^2+H-b\right] d\xi$$

$$= \frac{1}{3(1-b)} \int_{-\infty}^{\infty} (H-1)(H-b)(H+b+1)\, d\xi$$

don't see offhand why that is not in the answer as given.

integrand $(H-1)(H-b)(H+b[+1]) = p(H)$



has $p(H) < 0$ for $b < H < 1$

so if H is monotonic, $p(H) < 0$

$$\Rightarrow \lambda_1 < 0 \Rightarrow \text{stable}$$