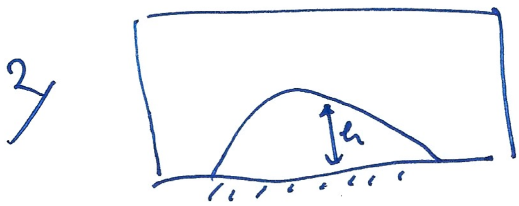
C(1

C5.7 Topics in fluids 2016 q2



(a) $\underline{u} = -\frac{k}{\mu}\left[\underline{\nabla} p + \rho g \underline{k}\right]$ 2-D

$\Rightarrow u = -\frac{k}{\mu} p_x$

$w = -\frac{k}{\mu}(p_z + \rho g)$

If $z \ll x$ at long times as blob spreads, $\frac{z}{x} \sim \delta$

$u_x + w_z = 0$ so $w \sim \delta u$

$\frac{\partial}{\partial x} \sim \delta \frac{\partial}{\partial z}$

So $u \sim \frac{k}{\mu} p_x \gg w$

$\wedge$

$\frac{k}{\mu} p_z$ $\Rightarrow w \ll \frac{k}{\mu} p_z$

So approx $p_z = -\rho g \Rightarrow p = \rho g(h - z)$

$u \simeq -\frac{k}{\mu}\rho g h_x$

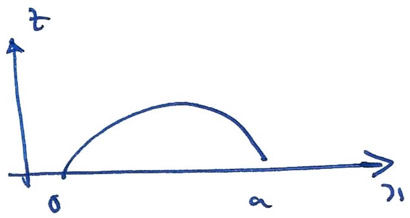
mass conservation $\phi h_t + \underline{\nabla} \cdot \int_0^h \underline{u}\, dz = 0$ in general

here $\phi h_t + \frac{\partial}{\partial x}\int_0^h u\, dz = 0$

$\phi h_t = -\frac{\partial}{\partial x}(hu) = \frac{k\rho g}{\mu}(h h_x)_x$

C2

(b)



$$W = \int_0^a x h \, dx$$

$$\frac{dW}{dt} = \dot a \, a h\big|_a + \int_0^a x h_t \, dx$$

$$= \int_0^a \frac{k\rho g}{2\phi\mu} x (h^2)_{xx} \, dx$$

$$= \frac{k\rho g}{2\phi\mu}\left[x(h^2)_x \Big|_0^a - \int_0^a (h^2)_x \, dx \right]$$

$$= \frac{k\rho g}{2\phi\mu}\left[- h^2\Big|_0^a \right]$$

$$= 0$$

$$x, h \sim W^{1/3} \qquad t \sim \frac{2\phi W^{1/3}\mu}{k\rho g}$$

$$\Rightarrow \quad \phi W^{1/3} \frac{k\rho g}{\mu \, 2\phi W^{1/3}} h_t = \frac{k\rho g}{2\mu} (h^2)_{xx}$$

$$\Rightarrow \quad h_t = (h^2)_{xx} \quad \& \quad \int_0^a x h \, dx = 1$$

(c) $h = t^\alpha H(\eta) \qquad \eta = \frac{x}{t^\beta} \qquad a = t^\beta \xi$

$$\Rightarrow \quad \alpha t^{\alpha-1} H - \beta\eta t^{\alpha-1} H' = t^{2\alpha-2\beta} (H^2)''$$

So $\alpha - 1 = 2\alpha - 2\beta$

Also $1 = \int_0^a x h \, dx = t^{2\beta+\alpha} \int_0^\xi \eta H \, d\eta$

$\Rightarrow \alpha + 2\beta = 0 \;\&\; \alpha = 2\beta - 1 \quad \Rightarrow \beta = \frac{1}{4} \quad \alpha = -\frac{1}{2}$

So $-\frac{1}{2}H - \frac{1}{4}\eta H' = (H^2)''$

& $\int_0^{\xi} \eta H \, d\eta = 1$, $H(0) = H(\xi) = 0$

$\times \eta$ $\quad -\frac{1}{2}\eta H - \frac{1}{4}\eta^2 H' = \eta (H^2)''$

$\Rightarrow -\frac{1}{4}(\eta^2 H)' = [\eta (H^2)']' - (H^2)'$

$\Rightarrow -\frac{1}{4}\eta^2 H = \eta (H^2)' - H^2$

$\Rightarrow -\frac{1}{4}\eta^2 = 2\eta H' - H$

$\Rightarrow \underbrace{\frac{2H'}{\sqrt{\eta}} - \frac{H}{\eta^{3/2}}}_{} = -\frac{1}{4}\eta^{1/2}$

$\left(\frac{2H}{\sqrt{\eta}}\right)' = -\frac{1}{4}\eta^{1/2}$

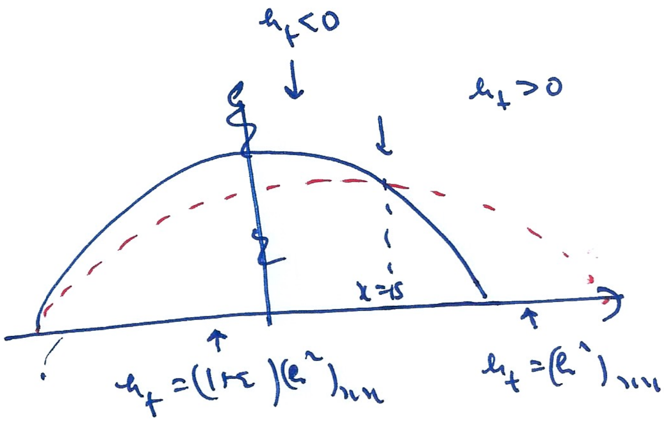
$\frac{2H}{\sqrt{\eta}} = \frac{1}{4}\cdot\frac{2}{3}\left(\xi^{3/2} - \eta^{3/2}\right)$ $\quad$ ($H = 0$ at $\eta = \xi$)

$H = \frac{1}{12}\left(\xi^{3/2}\eta^{1/2} - \eta^2\right)$

$\int_0^{\xi} \eta H \, d\eta = 1 \quad \Rightarrow \quad \frac{1}{12}\left(\frac{2}{3} - \frac{1}{3}\right)\xi^3 = 1$

$\Rightarrow \xi = 36^{1/3}$

(d)



$$w = \int x h \, dx = \int_0^s + \int_s^a x h \, dx$$

$$\frac{dw}{dt} = \frac{d}{dt}\int_0^s x h \, dx + \frac{d}{dt}\int_s^a x h \, dx$$

$$= (xh)\big|_s \dot{s} + \int_0^s x h_t - (xh)\big|_s \dot{s} + \int_s^a x h_t \, dx$$

$$= \int_0^s x(1+\varepsilon)(h^2)_{xx} \, dx + \int_s^a x(h^2)_{xx} \, dx$$

$$= (1+\varepsilon)\left[x(h^2)_x\big|_0^s - \int_0^s (h^2)_x \, dx \right] + x(h^2)_x\big|_s^a - \int_s^a (h^2)_x \, dx$$

$$= (1+\varepsilon)\, s(h^2)_x\big|_{s-} - (1+\varepsilon) h^2\big|_s - s(h^2)_x\big|_{s+} + h^2\big|_s$$

(h is cts at $x=s$)

(assuming h_x is cts at $x=s$)

$$= \varepsilon\left[s(h^2)_x\big|_s - h^2\big|_s \right]$$ as required

h_x being cts is a consequence of $[h(s(t),t)]_-^+ = 0$

$$\Rightarrow [h_t + \dot{s} h_x]_-^+ = 0$$

$$\Rightarrow [h_x]_-^+ = 0 \text{ if } \dot{s} \neq 0$$ since $h_t = 0$ at $x=s$.

Self-similar solution: also with $s = \sigma t^{\beta}$ $\quad x = t^{\beta}\eta$ $\quad h = t^{\alpha} H$

as before $\quad \alpha - 1 = 2\alpha - 2\beta \Rightarrow \alpha = 2\beta - 1 \Rightarrow \beta = \frac{\alpha+1}{2}$

$$\& \quad -\alpha H - \beta\eta H' = (1+\varepsilon)(H^2)'' , \quad \eta < \sigma$$
$$= (H^2)'' \qquad \eta > \sigma$$

with H, H' continuous at $\eta = \sigma$

and $(h_t = 0)$ $\quad \alpha H + \beta\eta H' = 0$ at $\eta = \sigma$

$$w = \int_0^a x h \, dx = t^{2\beta+\alpha} \int_0^{\zeta} \eta H \, d\eta$$

thus $\quad (\alpha + 2\beta) t^{\alpha+2\beta-1} \int_0^{\zeta} \eta H \, d\eta$

$$= \varepsilon \left[\sigma t^{\beta} t^{2\alpha-\beta} (H^2)' - t^{2\alpha} H^2 \right] \Big|_{\eta=\sigma}$$

~~now with $\beta = \alpha+1$~~ $\Rightarrow (\alpha + 2\beta)\, t^{\alpha+2\beta-1-2\alpha} \int_0^{\zeta} \eta H \, d\eta$

~~$\alpha + 2\beta - 1 =$~~ $\quad = \varepsilon \left[\sigma (H^2)' - H^2 \right] \Big|_{\eta=\sigma}$

viz $\quad (\alpha + 2\beta) \int_0^{\zeta} \eta H \, d\eta = \varepsilon \left[\sigma(H^2)' - H^2 \right]_{\eta=\sigma}$

~~this actually requires $\alpha = 2\beta - 1$??~~ $\quad$ ~~(? no need to satisfy this (or mistake in algebra?~~