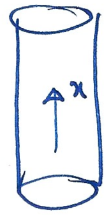
C5.7 Topics in fluids 2016 q3

1)



$\chi_g = 1 \quad x \in G$ (G gas phase)

$\ 0 \quad x \notin G$

Mass conservation $\quad \rho_{gt} + \nabla\cdot(\rho_g \mathbf{v}) = 0$

$\times \chi_g$ integrate over $A \times (x, x+\delta x)$:  V

$$\Rightarrow 0 = \int_V \chi_g\left[\rho_{gt} + \nabla\cdot(\rho_g \mathbf{v})\right] dV$$

$$= \int_V \left\{(\rho_g \chi_g)_t + \nabla\cdot\left[\chi_g \rho_g \mathbf{v}\right]\right\} dV - \int_V \left[\rho_g \frac{\partial \chi_g}{\partial t} + \rho_g \mathbf{v}\cdot\nabla \chi_g\right] dV$$

$$\Rightarrow \delta x \frac{\partial}{\partial t}\int_A \rho_g \chi_g \, dA + \left[\int_A \chi_g \rho_g v \, dA\right]_x^{x+\delta x} - \int_V \rho_g \frac{d\chi_g}{dt} dV$$

where $v = \mathbf{v}\cdot\mathbf{i}$ ($\mathbf{i}$ in x direction)

But $\frac{d\chi_g}{dt} = 0$ in the sense of generalised functions as $\forall \phi$ vanishing at ∞

$$\int_V\int_t \phi \frac{d\chi_g}{dt} dV dt = -\int_V\int_t \chi_g \frac{d\phi}{dt} dV dt$$

$$= -\int_t \int_G \frac{d\phi}{dt} dV dt$$

$$= -\int_t \frac{d}{dt}\int_G \phi \, dV dt \quad \text{by Reynolds' transport theorem}$$

$$= 0 \quad \text{as } \phi \to 0 \text{ at } \pm\infty.$$

$\div \delta x$ above $\Rightarrow \quad \frac{\partial}{\partial t}\int_A \rho_g \chi_g \, dA + \frac{\partial}{\partial x}\int_A \chi_g \rho_g v \, dA = 0$

(2

Define $\alpha = \frac{1}{A}\int x_s \, dA$

$$\alpha \bar{\rho}_s = \frac{1}{A}\int_A \rho_s x_s \, dA \quad \Rightarrow \bar{\rho}_s = \frac{\int_A \rho_s x_s \, dA}{\int_A x_s \, dA}$$

$$\alpha \bar{\rho}_s \bar{v} = \frac{1}{A}\int_A \rho_s x_s v \, dA \Rightarrow \bar{v} = \frac{\int_A \rho_s x_s v \, dA}{\int_A \rho_s x_s \, dA}$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\alpha \bar{\rho}_s\right) + \frac{\partial}{\partial x}\left(\alpha \bar{\rho}_s \bar{v}\right) = 0.$$

liquid $\quad \frac{\partial}{\partial t}\left[\bar{\rho}_l (1-\alpha)\right] + \frac{\partial}{\partial x}\left[(1-\alpha)\bar{\rho}_l \bar{u}\right] = 0$

$$\bar{\rho}_l = \frac{\int_A \rho_l (1-x_s) \, dA}{\int_A (1-x_s) \, dA}$$

$$\bar{u} = \frac{\int_A \rho_l (1-x_s) u \, dA}{\int_A \rho_l (1-x_s) \, dA}$$

Also $\alpha_t + (\alpha v)_x = 0$

$-\alpha_t + [(1-\alpha)u]_x = 0$

(b) $\rho_s (v_t + v v_x) = -p_x$

$\rho_l (u_t + u u_x) = -p_x$

(i) $\underline{\Psi} = \begin{pmatrix} \alpha \\ u \\ v \\ p \end{pmatrix}$

so $$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \rho_s & 0 \\ 0 & \rho_l & 0 & 0 \end{pmatrix}}_{A} \underline{\Psi}_t + \underbrace{\begin{pmatrix} v & 0 & \alpha & 0 \\ -u & 1-\alpha & 0 & 0 \\ 0 & 0 & \rho_s v & 1 \\ 0 & \rho_l u & 0 & 1 \end{pmatrix}}_{B} \underline{\Psi}_x = \underline{0}$$

Characteristics $\quad A\underline{\psi}_t + B\underline{\psi}_x = 0$

$$\frac{dx}{dt} = \lambda \quad , \quad \det(B - \lambda A) = 0$$

ie
$$\begin{vmatrix} v-\lambda & 0 & \alpha & 0 \\ \lambda-u & 1-\alpha & 0 & 0 \\ 0 & 0 & \rho_g(v-\lambda) & 1 \\ 0 & \rho_\ell(u-\lambda) & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (v-\lambda)(1-\alpha)\rho_g(v-\lambda) + \alpha(\lambda-u).-\rho_\ell(u-\lambda) = 0$$

$$\rho_g(1-\alpha)(v-\lambda)^2 \cancel{+} \rho_\ell\alpha(\lambda-u)^2 = 0$$

define $s = \left(\dfrac{\rho_g(1-\alpha)}{\rho_\ell\alpha}\right)^{\frac{1}{2}}$

$$\Rightarrow \quad s^2(\lambda-v)^2 = -(\lambda-u)^2$$

$$s(\lambda-v) = \pm i(\lambda-u)$$

$$(s \mp i)\lambda = sv \mp iu$$

$$\lambda = \frac{iu \mp sv}{i \mp s} \overset{\times -i}{=} \frac{u \pm isv}{1 \pm is}$$

ii $\quad u = v \Rightarrow \lambda_\pm = u$

$u \neq v \Rightarrow \lambda_\pm$ complex $\quad$ i.e. iff.

Linearise

$$\underline{\psi} = \underline{\psi}_0 + \underline{\Psi}$$

$$\Rightarrow \quad A_0 \underline{\Psi}_t + B_0 \underline{\Psi}_x = 0$$

A, B but ~~at~~ $u = u_0$ etc

Solutions $\quad \underline{\Psi} = e^{\sigma t + ikx} \underset{\sim}{w}$

$$\Rightarrow \quad \sigma A_0 \underline{w} + ik B_0 \underline{w} = 0$$

ie $\quad \left(B_0 - \left(\frac{-\sigma}{ik}\right) A_0\right) \underline{w} = 0$

$$\Rightarrow \quad -\frac{\sigma}{ik} = \lambda_\pm \text{ as above } = \frac{u_0 \pm i s v_0}{1 \pm is}$$

$$\Rightarrow \quad \sigma_\pm = \frac{-ik\left[u_0 \pm i s v_0\right]}{1 \pm is}$$

$$= -ik \frac{\left[u_0 \pm i s v_0\right]\left[1 \mp is\right]}{1 + s^2}$$

$$= \frac{-ik\left[u_0 + s^2 v_0 \pm is(v_0 - u_0)\right]}{1+s^2}$$

$$= \frac{\pm ks(v_0 - u_0) - ik(u_0 + s^2 v_0)}{1+s^2}$$

waves travel upwards (+ve x) & growth rate

$$\text{Re}\,\sigma = \pm \frac{s(v_0 - u)}{1+s^2} k$$

one is unstable $\text{Re}\,\sigma \to \infty$ as $k \to \infty$ - ill-posed

(c) $\overline{u^2} = D_\ell \bar{u}^2$

$$\bar{u} = \frac{\int_A \rho_\ell \chi_\ell u \, dA}{\int \rho_\ell \chi_\ell \, dA}$$

$$\overline{u^2} = \frac{\int_A \rho_\ell \chi_\ell u^2 \, dA}{\int \rho_\ell \chi_\ell \, dA}$$

~~Eg~~ (i) plug flow

If $u = u(x,t)$

then $\bar{u} = u$

$\overline{u^2} = u^2 = \bar{u}^2$ □.

(ii) one-phase

$u = c(1-r^2)$

So $\bar{u} = \frac{1}{A}\int u \, dA$

$\overline{u^2} = \frac{1}{A}\int u^2 dA$

so if $u = c(1-r^2)$

(radius 1) $\bar{u} = \frac{1}{\pi} \cdot 2\pi \int_0^1 c(1-r^2)\, r\, dr$

$= 2c\left[\frac{1}{2} - \frac{1}{4}\right] = \frac{c}{2}$

$\overline{u^2} = \frac{1}{\pi} \cdot 2\pi \int_0^1 c^2(1-r^2)^2 \, r\, dr$

$= 2c^2\left[\frac{1}{2} - \cancel{\frac{2}{4}}\frac{1}{2} + \frac{1}{6}\right]$

$= \frac{c^2}{3}$

$\overline{u^2} = \frac{1}{3}(2\bar{u})^2 = \frac{4}{3}\bar{u}^2 \Rightarrow \underline{D_\ell = 4/3}$