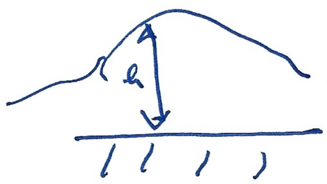
C5.7 Topics in fluids 2017 Q1

C1

1)



$h_t = (1+\varepsilon) h_{xx}$ $\quad h_t < 0$ $\quad h \to 0$ as $x \to \pm\infty$

$= h_{xx}$ $\quad h_t > 0$

(a) $\varepsilon = 0$ $\quad h_t = h_{xx}$ $\quad \frac{d}{dt}\int_{-\infty}^{\infty} h\,dx = \int_{-\infty}^{\infty} h_{xx}\,dx = [h_x]_{-\infty}^{\infty} = 0$

$\omega = \int_{-\infty}^{\infty} h\,dx$ is constant

$h = t^\alpha H(\eta)$ $\quad \eta = \frac{x}{t^\beta}$

$+\alpha t^{\alpha-1} H - \beta t^{\alpha-1}\eta H' = t^{\alpha-2\beta} H''$

$\Rightarrow$ $\beta = \frac{1}{2}$

$\omega = \int_{-\infty}^{\infty} t^{\alpha+\beta} H\,d\eta$ $\quad \Rightarrow \alpha+\beta = 0 \Rightarrow$ $\alpha = -\frac{1}{2}$

So $-\frac{1}{2}(H + \eta H') = H''$

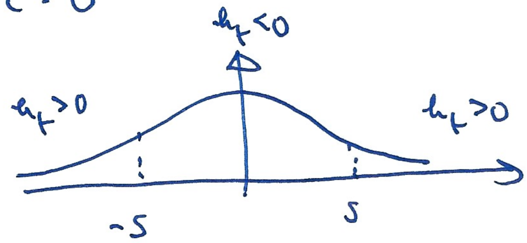
$H' = -\frac{1}{2}\eta H$

$H = A e^{-\frac{1}{4}\eta^2}$ $\quad \omega = \int_{-\infty}^{\infty} H\,d\eta = 2A\int_{-\infty}^{\infty} e^{-t^2}\,dt$

$= 2A\sqrt{\pi}$

$\Rightarrow A = \frac{\omega}{2\sqrt{\pi}}$, $\quad H = \frac{\omega}{2\sqrt{\pi}} e^{-\frac{1}{4}\eta^2}$

(b) $Q > 0$



$0<x<S \quad h_t = (1+\varepsilon) h_{xx}$

h symmetric

$x>S \quad h_t = h_{xx}$

$$w = 2\int_0^\infty h\,dx$$

$$\dot{w} = 2\left[\int_0^S + \int_S^\infty h_t\,dx\right]$$

$$= 2\left[\int_0^S (1+\varepsilon) h_{xx}\,dx + \int_S^\infty h_{xx}\,dx\right]$$

$$= 2\left[(1+\varepsilon) h_x|_S - h_x|_S\right]$$

$$= 2\varepsilon h_x|_S$$

As before, & $S = \sigma t^\beta$, $\beta = \frac{1}{2}$

$\Rightarrow \quad w = 2t^{\alpha+\frac{1}{2}} \int_0^\infty H\,d\eta$

& $\alpha H - \frac{1}{2}\eta H' = \begin{cases} H'' & \eta > \sigma \\ (1+\varepsilon) H'' & \eta < \sigma \end{cases}$, $\quad \alpha H - \frac{1}{2}\eta H' = 0$ at $\eta = \sigma$

So $\dot{w} = (2\alpha+1) t^{\alpha-\frac{1}{2}} \int_0^\infty H\,d\eta$

$$= 2\varepsilon t^{\alpha-\beta} H'(\sigma)$$

So $(2\alpha+1) \int_0^\infty H\,d\eta = 2\varepsilon H'(\sigma)$

(c) $\alpha = \alpha_0 + \varepsilon\alpha_1 + \ldots$ etc.

lead order as for (a) $\alpha_0 = -\frac{1}{2}$ $\quad H_0 = \dfrac{\omega}{2\sqrt{\pi}} e^{-\frac{1}{4}\eta^2}$

So σ_0 : $\quad -\frac{1}{2}(\eta H_0)' = 0$

$$\eta H_0 \propto \eta e^{-1/4\eta^2}$$

$$(\eta H_0)' \propto e^{-1/4\eta^2}\left[1 - \tfrac{1}{2}\eta^2\right] = 0$$

$$\eta = \sqrt{2}$$

So $\sigma_0 = \sqrt{2}$

& from $(2\alpha+1)\int_0^\infty H\,d\eta = 2\varepsilon H'(\sigma)$

we have $2\alpha_1 \int_0^\infty H_0\,d\eta = 2\,H_0'(\sigma_0)$

$$2\alpha_1\omega = 2 \cdot \frac{\omega}{2\sqrt{\pi}} \cdot -\tfrac{1}{2}\sigma_0 e^{-1/4\sigma_0^2}$$

$$\alpha_1 = -\frac{1}{4\sqrt{\pi}} \cdot \sqrt{2} \cdot e^{-1/2} = -\frac{1}{4}\sqrt{\frac{2}{\pi e}}$$