Qn C5.7 Topics in fluids 2017 q2

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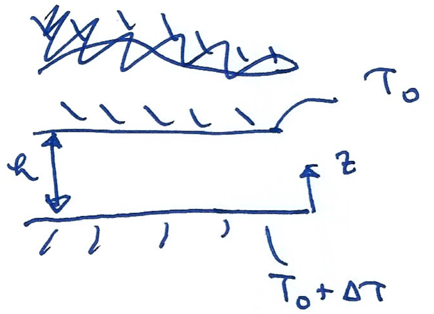


$$\begin{cases} \rho = \rho_0[1-\beta(T-T_0)] \\ \underline{u} = -\dfrac{k}{\mu}[\underline{\nabla}p + \rho g \underline{k}] \\ \underline{\nabla}\cdot\underline{u} = 0 \\ T_t + \underline{u}\cdot\underline{\nabla}T = \kappa_T \nabla^2 T \end{cases}$$

(a) $\partial_t = \underline{u} = 0 \Rightarrow T = T_0 + \dfrac{\Delta T}{h}(h-z)$

$$\rho = \rho_0\left[1 - \frac{\beta\Delta T}{h}(h-z)\right]$$

$$\frac{\partial p}{\partial z} = -\rho g = -\rho_0 g\left[1-\beta\frac{\Delta T}{h}(h-z)\right]$$

$$\Rightarrow p = p_h + \rho_0 g(h-z) - \tfrac{1}{2}\rho_0 g\beta\frac{\Delta T}{h}(h-z)^2 \qquad p_h \text{ constant}$$

(b)(i) Linearise $T = T_s + \theta$, $\rho = \rho_s + R$, $\underline{u}, \theta, R \ll 1$, $p = p_s + P$

$$\Rightarrow R = -\beta\rho_0\theta$$

$$\underline{u} = -\frac{k}{\mu}[\underline{\nabla}P + R g \underline{k}]$$

$$\underline{\nabla}\cdot\underline{u} = 0$$

$$\theta_t - \frac{\Delta T}{h}w = \kappa_T\nabla^2\theta \qquad w = \underline{u}\cdot\underline{k}$$

Non-d

$$\underset{\sim}{x} \sim h \qquad \underset{\sim}{u} \sim \frac{\kappa_T}{h} \qquad t \sim \frac{h^2}{\kappa_T}, \quad p \sim \frac{\mu \kappa_T}{\cancel{k}}, \quad \theta \sim \Delta T$$

$$\Rightarrow \quad \frac{\kappa_T}{h} \underset{\sim}{u} = -\frac{k}{\mu} \left[\frac{\mu \kappa_T}{k h} \underset{\sim}{\nabla} p - \beta \rho_0 g \, \theta \, \underset{\sim}{k} \right]$$

$$\Rightarrow \underset{\sim}{u} = -\underset{\sim}{\nabla} p + Ra\, \theta \underset{\sim}{k} \qquad Ra = \frac{\beta \rho_0 g k h}{\mu \kappa_T}$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{u} = 0$$

$$\frac{\kappa_T \Delta T}{h^2} \theta_t - \frac{\Delta T}{h} \frac{\kappa_T}{h} w = \frac{\kappa_T \Delta T}{h^2} \nabla^2 \theta$$

$$\Rightarrow \theta_t - w = \nabla^2 \theta$$

$$\text{curl } \underset{\sim}{u} = Ra \text{ curl } \theta \underset{\sim}{k}$$

$$\Rightarrow \text{curl curl } \underset{\sim}{u} = Ra \text{ curl curl } \theta \underset{\sim}{k}$$

$$\underbrace{\text{grad div } \underset{\sim}{u}}_{= 0} - \text{curl curl } \underset{\sim}{u} = - Ra \text{ curl curl } \theta \underset{\sim}{k}$$

$$\Rightarrow \nabla^2 \underset{\sim}{u} = Ra \nabla^2 \theta \underset{\sim}{k} - Ra \text{ grad div } \theta \underset{\sim}{k}$$

$$\text{div } \theta \underset{\sim}{k} = \underset{\sim}{\nabla} \theta \cdot \underset{\sim}{k} = \theta_z$$

$$\text{grad div } \theta \underset{\sim}{k} = + \frac{\partial}{\partial z} \underset{\sim}{\nabla} \theta$$

$$\text{So } \nabla^2 \underset{\sim}{u} = Ra \nabla^2 \theta \underset{\sim}{k} - Ra \left(\theta_{xz}, \theta_{yz}, \theta_{zz} \right)$$

$$\cdot \underset{\sim}{k}. \Rightarrow \nabla^2 w = Ra \nabla_H^2 \theta$$

(c) as now

$$\theta_t = \nabla^2 \theta + w$$

$$\frac{1}{Pr} \nabla^2 w_t = Ra \nabla_H^2 \theta + \nabla^4 w$$

Normal modes in horizontal $\propto e^{\sigma t + ikx + ily}$

for bounded solutions as $x \to \infty$ (or via Fourier transform)

constant coefficients so solutions exponential in z also

bc's at $z = 0, 1$: $w = w_{zz} = 0$, $\theta = 0$

& more $$\frac{1}{Pr}\left(\frac{\partial}{\partial t} - \nabla^2\right) \nabla^2 w_t = Ra \nabla_H^2 w + \left(\frac{\partial}{\partial t} - \nabla^2\right) \nabla^4 w$$

~~& $\theta = 0$ at $z = 0, 1 \Rightarrow$~~

$w = w_{zz} = 0$ on $z = 0, 1 \Rightarrow \nabla^2 w = 0$ there

So also $\theta = 0 \Rightarrow \nabla^4 w = 0$ there i.e. $w'''' = 0$

So by inspection of w eq + bc's $w = \sin m\pi z$ works, $m \in \mathbb{Z}$

& we require $\cancel{\frac{1}{Pr}(\sigma + K^2) \cdot -K^2 \sigma = -Ra K^2 + (\sigma + K^2) K}$

$$\frac{1}{Pr}\left(\sigma + K^2 + m^2\pi^2\right) \cdot -(K^2 + m^2\pi^2)\sigma = -Ra K^2 + (\sigma + K^2 + m^2)(K^2 + m^2)^2$$

where $K^2 = k^2 + l^2$

viz $$\frac{1}{Pr}(K^2 + m^2\pi^2)(\sigma + K^2 + m^2\pi^2)\sigma + (\sigma + K^2 + m^2)(K^2 + m^2)^2 - Ra K^2 = 0$$

(could just assume exchange of stability at $\sigma = 0$ but in more detail:)

Let's write $J^2 = k^2 + m^2\pi^2$

~~so~~

$$\cancel{\frac{1}{Pr}(k^2+m^2\pi^2)(\sigma+k^2+m^2\pi^2)}$$

$$\frac{1}{Pr} J^2 (\sigma + J^2)(\sigma + J^2) - \frac{1}{Pr} J^4 (\sigma + J^2)$$

$$+ J^4(\sigma + J^2) - Ra\, k^2 = 0$$

$$\frac{1}{Pr} J^2 (\sigma + J^2)^2 + J^4\left(1 - \frac{1}{Pr}\right)(\sigma + J^2) - Ra\, k^2 = 0$$

discriminant always +ve so real roots

If $Ra \to 0$ roots are approx.

$$\sigma = -J^2 < 0$$

and $$\sigma + J^2 = \frac{-J^4\left(1 - \frac{1}{Pr}\right)}{\frac{1}{Pr} J^2} = -J^2(Pr - 1)$$

i.e. $\sigma = -Pr\, J^2 < 0$

Therefore instability occurs at Ra^* where $\sigma = 0$ (assuming $\frac{d\sigma}{dRa} > 0$ there

– transversality : $\frac{2}{Pr} J^2 (\sigma + J^2)\sigma' + J^4\left(1 - \frac{1}{Pr}\right)\sigma' = k^2$

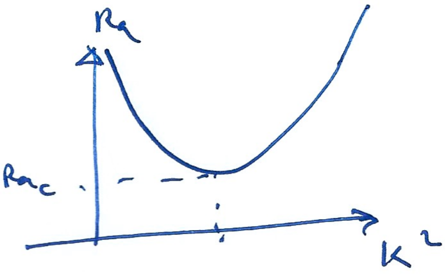
at $\sigma = 0$ $J^4 \sigma' \left(\frac{2}{Pr} + 1 - \frac{1}{Pr}\right) = k^2 \Rightarrow \sigma' > 0$)

At $\sigma = 0$ $\quad \frac{J^6}{Pr} + J^6\left(1 - \frac{1}{Pr}\right) = Ra\, K^2$

$$\Rightarrow \quad Ra = \frac{J^6}{K^2} = \frac{(K^2 + m^2\pi^2)^3}{K^2}$$

minimum such Ra is if $m = 1$

$$Ra = \frac{(K^2 + \pi^2)^3}{K^2}$$



critical value at min:

$$\frac{3(K^2+\pi^2)^2}{K^2} - \frac{(K^2+\pi^2)^3}{K^4} = 0$$

$$3K^2 \cancel{K} = K^2 + \pi^2 \qquad K^2 = \frac{\pi^2}{2}$$

$$K = \frac{\pi}{\sqrt{2}}$$

$$\& \; Ra_c = \frac{\left(\frac{3\pi^2}{2}\right)^3}{\frac{\pi^2}{2}} = \frac{27\pi^4}{4}$$