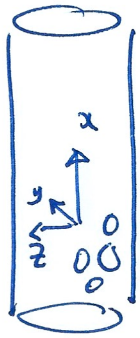
CS.7 topics in fluids answer 2017 q3

3(a)




The fluid consists of gas G & liquid L (subsets of $\mathbb{R}^3$)

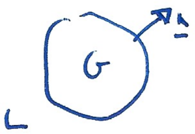
The indicator function $\chi_G = 1$ if $\underline{x} \in G$

$= 0$ if $\underline{x} \in L$

Mass of gas $\qquad \frac{\partial \rho_g}{\partial t} + \underline{\nabla} \cdot [\rho_g \underline{v}] = 0$

$\times \chi_g$ integrate over cylindrical volume $V$ : 


$$\Rightarrow \int_V \left[ \chi_g \frac{\partial \rho_g}{\partial t} + \chi_g \underline{\nabla} \cdot [\rho_g \underline{v}] \right] dV = 0$$




$$\Rightarrow \int_V \left[ \frac{\partial}{\partial t}(\chi_g \rho_g) + \underline{\nabla} \cdot [\chi_g \rho_g \underline{v}] \right] dV$$

$$= - \int_V \left[ \rho_g \frac{\partial \chi_g}{\partial t} + \rho_g \underline{v} \cdot \underline{\nabla} \chi_g \right] dV$$

treating $\chi_g$ as a generalised function

$$\Rightarrow \frac{\partial}{\partial t} \int_V \rho_g \chi_g \, dV + \left[ \int_A \rho_g \chi_g v \, dA \right]_-^+ = - \int_V \rho_g \frac{d\chi_g}{dt} \, dV$$

where $A$ is cross-section area & $v = \underline{v} \cdot \underline{n} =$ $\underline{v} \cdot \underline{i}$ ($\underline{i}$ in $x$ direction)

(since $\underline{v} \cdot \underline{n} = 0$ on walls) $[\;]_-^+ =$ jump from bottom to top

Now the definition of $\frac{d\chi_g}{dt} = \frac{\partial \chi_g}{\partial t} + \underline{v}\cdot\underline{\nabla}\chi_g$ as a generalised function is via integration over $\mathbb{R}^3 \times \mathbb{R}$ with a test function $\phi$ ($\to 0$ at $\infty$)

$$\iint_{\mathbb{R}^3\,\mathbb{R}} \phi\left[\frac{\partial \chi_g}{\partial t} + \underline{v}\cdot\underline{\nabla}\chi_g\right] dV\, dt$$

$$= -\iint \chi_g\left[\frac{\partial \phi}{\partial t} + \underline{\nabla}\cdot(\phi\underline{v})\right] dV\, dt$$

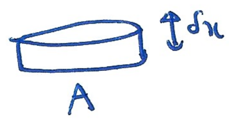
$$= -\int_{-\infty}^{\infty}\int_G \left[\frac{\partial \phi}{\partial t} + \underline{\nabla}\cdot(\phi\underline{v})\right] dV\, dt$$

$$= -\int_{-\infty}^{\infty}\left\{\frac{d}{dt}\int_G \phi\, dV\right\} dt$$ via Reynolds' transport theorem (material $G$)

$$= -\int_G \phi\, dV\Big|_{t=-\infty}^{t=\infty} = 0$$ as $\phi \to 0$ at $t \to \pm\infty$.

so $\frac{d\chi_g}{dt} = 0$ in the sense of generalised functions

Now let
$V = A\,\delta x$ :




$$\frac{\partial}{\partial t}\int_A \rho_g \chi_g\, dA\, \delta x + \delta\int_A \rho_g \chi_g v\, dA = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\int_A \rho_g \chi_g\, dA + \frac{\partial}{\partial x}\int_A \rho_g \chi_g v\, dA = 0$$

We define the gas volume fraction $\alpha$, average cross-section gas density $\bar{\rho}_g$, average cross-section gas velocity $\bar{v}$, via

$$\alpha = \frac{1}{A}\int_A \chi_g\, dA$$

$$\alpha\bar{\rho}_g = \frac{1}{A}\int_A \rho_g \chi_g\, dA$$

$$\alpha\bar{\rho}_g\bar{v} = \frac{1}{A}\int_A \rho_g \chi_g v\, dA$$

in other words

$$\bar{\rho}_S = \frac{\int_A \rho_S \chi_S \, dA}{\int_A \chi_S \, dA}$$

$$\bar{v} = \frac{\int \rho_S \chi_S v \, dA}{\int \rho_S \chi_S \, dA}$$

For liquid phase, $\chi_\ell$ is indicator function $= 1$ if $\underline{x} \in L$ (liquid)
$= 0$ if $\underline{x} \in G$

then
$$\bar{\rho}_\ell = \frac{\int_A \rho_\ell \chi_\ell \, dA}{\int_A \chi_\ell \, dA}$$

$$\bar{u} = \frac{\int \rho_\ell \chi_\ell v \, dA}{\int \rho_\ell \chi_\ell \, dA} \qquad (v = \underline{v} \cdot \underline{i} \text{ as before})$$

$$\Rightarrow \frac{\partial}{\partial t}\left[\bar{\rho}_\ell (1-\alpha)\right] + \frac{\partial}{\partial x}\left[\bar{\rho}_\ell (1-\alpha)\bar{u}\right] = 0$$

[illegible]

(iv)

(drop the bars), $\rho_s$, $\rho_l$ constant

$$\Rightarrow \quad \alpha_t + (\alpha v)_x = 0$$

$$-\alpha_t + [(1-\alpha)u]_x = 0$$

& then

$$\rho_s \alpha (v_t + v v_x) = -\alpha p_x - \frac{F_{sl}}{A}$$

$$\rho_l \left[ (1-\alpha)(u_t + u u_x) + (D_l - 1)\{(1-\alpha)u^2\}_x \right]$$

$$= -(1-\alpha) p_x \quad \text{~~} \frac{(F_{ls}+F_{lw})}{A} \text{~~}$$

$$- \frac{(F_{lw} - F_{sl})}{A}$$

Scale $\alpha = 1 - B\beta$, $B = \frac{f_{lw}}{f_{sl}}$

$u \sim U$, $v \sim V$, $p - p_a \sim P$

$V = \alpha_0 v_0$, $P = \rho_s V^2$, $U = \varepsilon V$, $\varepsilon = \left( \frac{\rho_s f_{sl}}{\rho_l f_{lw}} \right)^{1/2}$

$x \sim \frac{R}{f_{sl}}$, $t \sim \frac{R}{f_{sl} U}$, note $t \sim \frac{x}{U}$

& $F_{sl} = 2\pi R \text{~~}\alpha\text{~~} \rho_s f_{sl} |v-u|(v-u)$

$= 2\pi R \rho_s f_{sl} V^2 \Phi_{sl}$, $\Phi_{sl} = (1 - B\beta)^{1/2} |v - \varepsilon u|(v - \varepsilon u)$
↑ non-d

$F_{lw} = 2\pi R \rho_l f_{lw} U^2 |u| u$

$= 2\pi R \rho_l f_{lw} V^2 \varepsilon^2 \Phi_{lw}$, $\Phi_{lw} = |u| u$ ← non-d

So we get $\left(t \sim \frac{x}{U}\right)$

$$-\varepsilon B\beta_t + [(1-B\beta)v]_x = 0$$

$$B\beta_t + [B\beta u]_x = 0$$

$$\Rightarrow \quad \beta_t + (\beta u)_x = 0$$

Also (write $l = \frac{R}{f_{sl}} \sim x$) so $t \sim \frac{l}{U}$

$$\rho_s[1-B\beta]\left[\frac{UV}{l}v_t + \frac{V^2}{l}vv_x\right] = -(1-B\beta)\frac{\rho_s V^2}{l}p_x - \frac{2\pi R \rho_s f_{sl} V^2 \Phi_{sl}}{A}$$

(where $A = \pi R^2$)

$\div \frac{\rho_s V^2}{l}$ $\Rightarrow$ $(1-B\beta)[\varepsilon v_t + vv_x] = -(1-B\beta)p_x$

$$- \underbrace{\frac{R}{f_{sl}\rho_s V^2} \cdot \frac{2\pi R \rho_s f_{sl} V^2}{\pi R^2}}_{2} \Phi_{sl}$$

viz $(1-B\beta)(\varepsilon v_t + vv_x) = -(1-B\beta)p_x - 2(1-B\beta)^{?}|v-\varepsilon u|(v-\varepsilon u)$

and finally

$$\rho_\ell \left[ B\beta \frac{U^2}{\ell}(u_t + u u_x) + \frac{U^2}{\ell}(D_\ell - 1)\{B\beta u^2\}_x \right]$$

$$= -B\beta \frac{\rho_s V^2}{\ell} p_x$$

$$- \frac{\left[ 2\pi R \overbrace{\rho_\ell f_{\ell w} V^2 \varepsilon^2}^{\rho_s f_{s\ell} V^2} \Phi_{\ell w} - 2\pi R \rho_s f_{s\ell} V^2 \Phi_{s\ell} \right]}{\pi R^2}$$

$$\underbrace{\qquad\qquad\qquad}_{-\frac{2}{R} \rho_s f_{s\ell} V^2 (\Phi_{\ell w} - \Phi_{s\ell})}$$

note $\varepsilon^2 = \dfrac{\rho_s f_{s\ell}}{\rho_\ell f_{\ell w}}$

$\div \dfrac{\rho_\ell U^2}{\ell}$ :

$$B\beta(u_t + u u_x) + (D_\ell - 1)\{B\beta u^2\}_x$$

$$= -B\beta \frac{\rho_s}{\rho_\ell \varepsilon^2} p_x - \frac{2}{R} \frac{\rho_s f_{s\ell} V^2}{\rho_\ell U^2} \frac{R}{f_{s\ell}} (\Phi_{\ell w} - \Phi_{s\ell})$$

note $B = \dfrac{f_{\ell w}}{f_{s\ell}}$

$\div B$

$$\Rightarrow \beta(u_t + u u_x) + (D_\ell - 1)(\beta u^2)_x$$

$$= -\frac{\rho_s}{\rho_\ell \varepsilon^2} \beta p_x - \frac{2 \rho_s f_{s\ell}}{\rho_\ell f_{s\ell} \varepsilon^2} (\Phi_{\ell w} - \Phi_{s\ell})$$

$\varepsilon^2 = \dfrac{\rho_s}{\rho_\ell B}$ $\Rightarrow$ $= -B\beta p_x - 2B(\Phi_{\ell w} - \Phi_{s\ell})$

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viz

$$\beta(u_t + uu_x) + (D_\ell - 1)(\beta u^2)_x$$
$$= -B\left[\beta p_x + 2\left\{|u|u - (1-B\beta)^{1/2}|v-\varepsilon u|(v-\varepsilon u)\right\}\right]$$

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inlet values

(**) $1 - B\beta_0 = \alpha_0, \quad Uu_0 = \bar{u}_0, \quad Vv_0 = \bar{v}_0, \quad p_a + Pp_0 = \bar{p}_0$

(c) $D_\ell - 1 \ll 1, \quad B \ll 1, \quad \varepsilon \ll 1$

$\Rightarrow \quad u_t + uu_x \simeq 0$

$vv_x = -p_x - 2|v|v$

$v_x = 0$

$\beta_t + (\beta u)_x = 0$

$x = 0 \quad \beta = \beta_0, \; v = 1, \; u = u_0 \quad \Rightarrow v = 1$

$(p_x = -2 \quad p = p_0 - 2x)$

By inspection, or using characteristics, $u \equiv u_0$ & $\beta \equiv \beta_0$