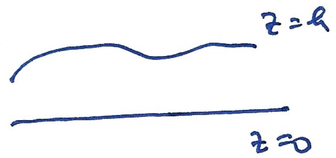
C5.7 Topics in fluids 2018 Q1

1.



$$u_x + w_z = 0$$

$$\rho[u_t + u u_x + w u_z] = -p_x + \mu \nabla^2 u$$

$$\rho[w_t + u w_x + w w_z] = -p_z + \mu \nabla^2 w$$

bcs $z = 0$ $u = w = 0$

$z = h$: $h_t + u h_x - w = 0$

$$-(p - p_0) + \frac{2\mu}{1 + h_x^2}\left[h_x^2 u_x - h_x(u_z + w_x) + w_z\right] = \frac{\gamma h_{xx}}{(1 + h_x^2)^{3/2}}$$

$$2h_x(u_x - w_z) + (h_x^2 - 1)(u_z + w_x) = 0$$

$z \sim H$ $x \sim L$, $u \sim U$ to be determined $\delta = \frac{H}{L}$, $w \sim \delta U$

$$p - p_0 \sim \frac{\mu U L}{H^2}, \quad t \sim \frac{L}{U}$$

Leads to non-d

$$u_x + w_z = 0$$

$$\frac{\rho U^2}{L}[u_t + u u_x + w u_z] = \frac{\mu U}{H^2}[-p_x + u_{zz} + \delta^2 u_{xx}]$$

$$\frac{\delta \rho U^2}{L}[w_t + u w_x + w w_z] = \frac{\mu U}{\delta H^2}[-p_z + \delta^2 w_{zz} + \delta^4 w_{xx}]$$

Define a Reynolds number $Re = \frac{\rho U H}{\mu}$ (for example)

$\frac{\rho U^2}{L} \cdot \frac{H^2}{\mu U} = \frac{\rho U H}{\mu} \cdot \delta$ then

$$\left.\begin{aligned} \delta Re[u_t + u u_x + w u_z] &= -p_x + u_{zz} + \delta^2 u_{xx} \\ \delta^3 Re[w_t + u w_x + w w_z] &= -p_z + \delta^2 w_{zz} + \delta^4 w_{xx} \end{aligned}\right\}$$

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Assuming $\delta \ll 1$ and $Re \lesssim 1$

we have $\quad p_x \simeq u_{zz}$

$\qquad\qquad\quad p_z = 0$.

B.C's. Integrated conservation of mass is (2-D)

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_0^h u\,dz = 0 \qquad \text{(dimensionally, or non-d)}$$

Stress condition on $z = h$ (non-d)

$$-\frac{\mu U L}{H^2}\,p + \cancel{\ }\frac{2\mu}{1+\delta^2 h_x^2}\left[\frac{H^2}{L^3}U\,h_x^2 u_x - \delta\frac{U}{H}(u_z+\delta^2 w_x) + \frac{\delta U}{H}w_z\right] = +\frac{\gamma H}{L^2}\,\frac{h_{xx}}{(1+\delta^2 h_x^2)^{3/2}}$$

or

$$-p + \frac{H^2}{\mu U L}\cdot\mu\cdot\frac{\delta U}{H}\cdot\frac{2}{1+\delta^2 h_x^2}\left[w_z - (u_z+\delta^2 w_x) + \delta^2 h_x^2 u_x\right] = +\frac{\gamma H}{L^2}\cdot\frac{H^2}{\mu U L}\,\frac{h_{xx}}{(1+\delta^2 h_x^2)^{3/2}}$$

We define the Capillary number as

$$Ca = \frac{\mu U}{\gamma}$$

so

$$-p + \frac{2\delta^2}{1+\delta^2 h_x^2}\left[w_z - (u_z+\delta^2 w_x) + \delta^2 h_x^2 u_x\right] = +\frac{\delta^3}{Ca}\,\frac{h_{xx}}{(1+\delta^2 h_x^2)^{3/2}}$$

and as $\delta \to 0$, but retaining the capillary term

$$-p = \frac{\delta^3}{Ca} h_{xx} \quad \text{on } z = h.$$

Other condition is

$$2\delta^2 \frac{U}{H} h_x (u_x - w_z) + (\delta^2 h_x^2 - 1) \frac{U}{H} (u_z + \delta^2 w_x) = 0$$

$$\Rightarrow u_z \approx 0$$

Now $p_z = 0 \Rightarrow \quad p = -\frac{\delta^3}{Ca} h_{xx}$

$$u_{zz} = p_x = -\frac{\delta^3}{Ca} h_{xxx}$$

$$u_z = +\frac{\delta^3}{Ca} h_{xxx} (h - z)$$

$$u = +\frac{\delta^3}{Ca} h_{xxx} (hz - \tfrac{1}{2} z^2)$$

$$\int_0^h u \, dz = +\frac{\delta^3}{Ca} h_{xxx} \cdot \tfrac{1}{3} h^3$$

$$\Rightarrow \quad h_t + \frac{\delta^3}{3Ca} \frac{\partial}{\partial x} \left[h^3 h_{xxx} \right] = 0$$

Finally, we choose U so that $Ca = \frac{1}{3}\delta^3$; i.e. $U = \dfrac{\gamma H^3}{3\mu L^3}$

$$\Rightarrow h_t + [h^3 h_{xxx}]_x = 0$$

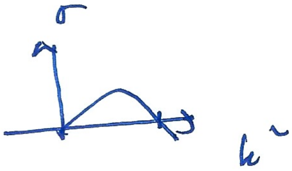
(b) vdW

$$\Rightarrow h_t + \left(h^3 h_{xxx} + \frac{a h_x}{h}\right)_x = 0$$

Linearise $h = 1 + \eta$

$$\eta_t + \eta_{xxxx} + a\eta_{xx} = 0$$

$\eta = e^{\sigma t + ikx} \Rightarrow \sigma = ak^2 - k^4$



unstable $k^2 < a$, $k < \sqrt{a}$

most unstable max σ at $a = 2k^2$ $\Rightarrow k = \sqrt{\frac{a}{2}}$

(c) $a = 1$ $\quad h_t + \left(h^3 h_{xxx} + \frac{h_x}{h}\right)_x = 0$

$h = (t_R - t)^\alpha H(\eta)$, $\eta = \frac{x}{(t_R - t)^\beta}$

$$\Rightarrow -\alpha(t_R - t)^{\alpha-1} H + \beta(t_R - t)^{\alpha-1}\eta H' + (t_R - t)^{4\alpha - 4\beta}(H^3 H''')' + (t_R - t)^{-2\beta}\left(\frac{H'}{H}\right)' = 0$$

$\Rightarrow \alpha - 1 = 4\alpha - 4\beta = -2\beta$

$\Rightarrow \beta = 2\alpha$

$\alpha - 1 = -2\beta = -4\alpha \qquad \alpha = \frac{1}{5}$, $\beta = \frac{2}{5}$

$$\Rightarrow -\tfrac{1}{5}H + \tfrac{2}{5}\eta H' + (H^3 H''')' + \left(\frac{H'}{H}\right)' = 0$$

~~Explain~~ .. it's not release of a point source – it is the approach to pinch-out of a film ($h \to 0$) at t_R.