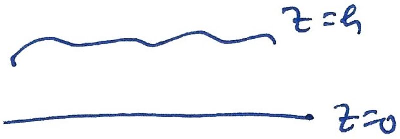
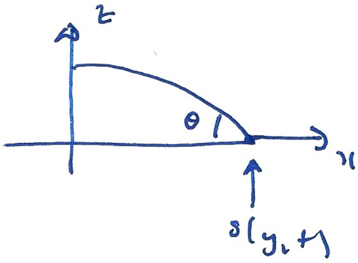
C5.7 Topics in fluids 2018 q 2 (more lubrication !!) (1

2)





h : periodic in y, symmetric in x

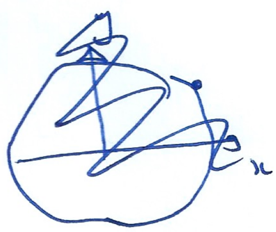
$$\mu h_t + \gamma \underline{\nabla}\cdot\left[(h^3 + \lambda h^2)\underline{\nabla}\nabla^2 h \right] = 0 \qquad \underline{\nabla} = \underline{\nabla}_H = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

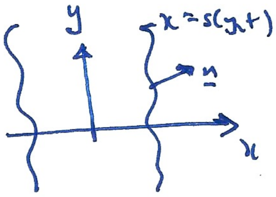
$x = s$: $h = 0$

$$v = -K\left[\left(\frac{\partial h}{\partial n}\right)^3 + \theta^3\right] \qquad v = \frac{s_t}{\sqrt{1+s_y^2}}$$

$$(h^3 + \lambda h^2)\frac{\partial}{\partial n}\nabla^2 h = 0$$

$x = 0$ $h_x = h_{xxx} = 0$





(a) Non-zero slip length is necessary to avoid infinite force (non-integrable ∞ stress) at contact line due to no-slip.

Scale $h, z \sim \theta L$, $x, y, s \sim L$, $t \sim \frac{L}{\theta^3 K}$, $v \sim \theta^3 K$

So θ is aspect ratio

$$\Rightarrow \quad \mu \theta L \frac{\theta^3 K}{L} h_t + \frac{\gamma}{L^4}\theta^4 L^4 \underline{\nabla}\cdot\left[\left(h^3 + \frac{\lambda}{\theta L}h^2\right)\underline{\nabla}\nabla^2 h\right] = 0$$

$$\Rightarrow \quad \frac{\mu K}{\gamma} h_t + \underline{\nabla}\cdot\left[(h^3 + \Lambda h^2)\underline{\nabla}\nabla^2 h\right] = 0$$

$$Ca = \frac{\mu K}{\gamma} \qquad \Lambda = \frac{\lambda}{\theta L}$$

Boundary conditions (cont-d)

$x=0 \quad h_x = h_{max} = 0$

$x = s : \quad h = 0$

$$v = \frac{s_t}{\sqrt{1+\theta^2 s_y^2}} = -\frac{K}{\theta^3 K}\left[\theta^3\left(\frac{\partial h}{\partial n}\right)^3 + \theta^3\right]$$

$$= -\left[\left(\frac{\partial h}{\partial n}\right)^3 + 1\right]$$

$$(h^3 + \Lambda h^2)\frac{\partial}{\partial n}\nabla^2 h = 0$$

(b) Steady state $\frac{\partial}{\partial y} = 0 \quad h = h(x)$

$\Rightarrow (h^3 + \Lambda h^2) h''' = 0 \qquad (h''' = 0 \text{ at } x = 0)$

Also need $\begin{cases} h' = -1 \text{ at } x = s, \; h' = 0 \text{ at } x = 0 \\ h = 0 \end{cases}$

$h'' = -A$

$h' = -Ax = -\frac{x}{s}$

$h = \frac{1}{2}\left[1 - \frac{x^2}{s^2}\right] = \frac{1}{2}(1 - x^2)$ if $s = 1$.

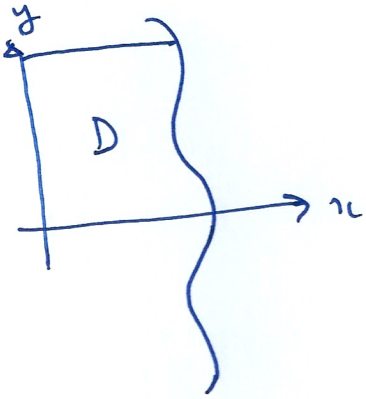
(c) $Ca \ll 1 \quad \Rightarrow \underline{\nabla}\cdot\left[(h^3 + \Lambda h^2)\underline{\nabla}\nabla^2 h\right] \approx 0$

$$\Rightarrow \int_D \nabla^2 h \; \underline{\nabla}\cdot\left[(h^3 + \Lambda h^2)\underline{\nabla}\nabla^2 h\right] dS = 0$$

ie $\int_D \left\{ \underline{\nabla} \cdot \left[\nabla^2 h \, (h^3 + \lambda h^2) \, \underline{\nabla}(\nabla^2 h) \right] - (h^3 + \lambda h^2) |\underline{\nabla} \nabla^2 h|^2 \right\} dS = 0$

Let D be $[0, s] \times [0, Y]$ if Y is period in y

ie



$$\Rightarrow 0 = \int_{\partial D} \nabla^2 h \, (h^3 + \lambda h^2) \frac{\partial}{\partial n} \nabla^2 h \, ds - \int_D (h^3 + \lambda h^2) |\underline{\nabla} \nabla^2 h|^2 \, dS$$

& ∂D integral is zero

$\Rightarrow \underline{\nabla} \nabla^2 h = 0$

$\Rightarrow \nabla^2 h = a(t)$

$x = 0 : h_x = 0$

$x = s : h = 0,$

$$\frac{s_t}{\sqrt{1 + 0^2 s_y^2}} = -\left[\left(\frac{\partial h}{\partial n} \right)^3 + 1 \right]$$

$$\underline{n} = \frac{(1, -s_y)}{\sqrt{1 + 0^2 s_y^2}}$$

(d)

$s = 1 + S$ (excuse notation)

$h = \frac{1}{2}(1-x^2) + H$

linearise ~~...~~ (note $a = -1 + A$)

$$\Rightarrow \quad \nabla^2 H = A \qquad x=0 \quad H_x = 0$$

$$\left[\tfrac{1}{2}(1-x^2) + H\right]\Big|_{x=1+S} = 0$$

$$\Rightarrow -S + H|_{x=1} \approx 0$$

$$\underset{\sim}{n} \simeq (1, -S_y)$$

$$\underset{\sim}{n} \cdot \underset{\sim}{\nabla} h = (1, -S_y) \cdot (-x\underline{i} + \underline{\nabla} H)$$

$$= (1, -S_y) \cdot (-x + H_x, H_y)$$

$$= -x + H_x - S_y H_y$$

$$\simeq -1 - S + H_x \quad \text{linearise}$$

so $$\frac{s_t}{\sqrt{1+0^2 S_y^2}} \simeq S_t \simeq -\left[(-1-S+H_x)^3 + 1\right]$$

$$\simeq -\left[-1 + 3.1.(-S+H_x) + 1\right]$$

$$\simeq 3(S - H_x)$$

So $\nabla^2 H = A \qquad x=0 \quad H_x = 0$

$$x = 1 \begin{cases} H = S \\ S_t = 3(S - H_x) \end{cases}$$

Put $H = \hat{h} e^{\sigma t + iky}$

$S = \hat{s} e^{\sigma t + iky}$

& we need $A = \hat{a} e^{\sigma t + iky}$ ~~...~~ $\hat{a} = 0$ [~~...~~]

– but $A = A(t)$ so require $\hat{a} = 0$ if $k \neq 0$

$$\Rightarrow \hat{h}'' - k^2 \hat{h} = 0$$

$x = 0 \quad \hat{h}' = 0$

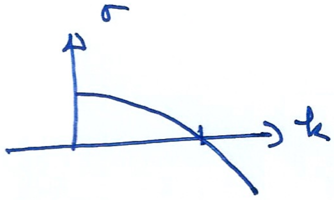
$x = 1 \quad \hat{h} = \hat{s}$

$\sigma \hat{s} = 3(\hat{s} - \hat{h}') \quad \Rightarrow \quad \sigma \hat{h} = 3(\hat{h} - \hat{h}')$

$$\Rightarrow h = \cosh kx$$

$$(\sigma - 3)\cosh k = -3k \sinh k$$

$$\sigma = 3\left[1 - k \tanh k\right]$$



Unstable for $k \tanh k < 1$