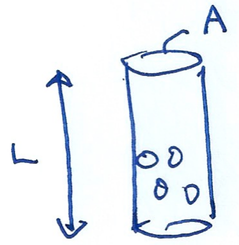
CS.7 Topics in fluids 2018 q3

I've gone a bit overboard here!

3(a)



Pointwise momentum equation is

$$\frac{\partial}{\partial t}(\rho_g \underset{\sim}{v}) + \underline{\nabla}\cdot[\rho_g \underline{v}\,\underline{v}] = \underline{\nabla}\cdot\underset{\sim}{\sigma}_g - \rho_g g\, \underline{i}$$

include gravity though not stated in question till (b)

$\underset{\sim}{\sigma}_g$ is gas stress tensor

The tensor $(\underline{v}\,\underline{v})_{ij} = v_i v_j$

$(\underline{\nabla}\cdot\underline{\sigma})_i = \dfrac{\partial \sigma_{ij}}{\partial x_j}$ for a tensor $\underline{\sigma}$

Also we'll need $\underline{\nabla}\cdot(\phi\underline{\sigma}) = \phi\,\underline{\nabla}\cdot\underline{\sigma} + \underline{\sigma}\cdot\underline{\nabla}\phi$

via $[\underline{\nabla}\cdot(\phi\underline{\sigma})]_i = \dfrac{\partial}{\partial x_j}\phi\sigma_{ij} = \phi\dfrac{\partial \sigma_{ij}}{\partial x_j} + \sigma_{ij}\dfrac{\partial \phi}{\partial x_j}$

Define gas volume as G, $X_g = 1 \quad \underline{x}\in G$
$\qquad = 0 \quad \underline{x}\notin G$

$$\int_V X_g\left[\frac{\partial}{\partial t}(\rho_g \underline{v}) + \underline{\nabla}\cdot\{\rho_g \underline{v}\,\underline{v}\}\right] dV = \int_V X_g\, \underline{\nabla}\cdot\underset{\sim}{\sigma}_g\, dV - \int_V \rho_g g X_g\, \underline{i}\, dV$$

where V is $A \times (x, x+\delta x)$:

$\{(\rho_g \underline{v}\,\underline{v})\cdot\underline{\nabla}X_g\}_i = \rho_g v_i v_j \dfrac{\partial X_g}{\partial x_j}$

$$\Rightarrow \int_V \left[\frac{\partial}{\partial t}(X_g \rho_g \underset{\sim}{v}) + \underline{\nabla}\cdot\{X_g \rho_g \underline{v}\,\underline{v}\}\right] dV$$
$$- \int_V \left[\rho_g \underset{\sim}{v}\frac{\partial X_g}{\partial t} + \rho_g \underset{\sim}{v}(\underline{v}\cdot\underline{\nabla}X_g)\right] dV$$
$$= \int_V \underline{\nabla}\cdot(X_g \underline{\sigma}_g)\, dV - \int_V \underline{\sigma}_g\cdot\underline{\nabla}X_g\, dV - \int_V \rho_g g X_g\, \underline{i}\, dV$$

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Now $\dfrac{dX_g}{dt} = 0$ in the sense of generalised functions

Since $\forall$ test functions $\phi \to 0$ at ∞ in $\underline{x}$ & t

$$\iint_{\mathbb{R}^3 \mathbb{R}} \phi \left[\frac{\partial X_g}{\partial t} + \underline{v}\cdot\underline{\nabla} X_g \right] dV\,dt$$

$$= -\iint X_g \left[\phi_t + \underline{\nabla}\cdot(\phi \underline{v}) \right] dV\,dt$$

$$= -\int_{-\infty}^{\infty} \left\{ \frac{d}{dt} \int_G \phi \, dV \right\} dt \qquad \text{via Reynolds' transport theorem (for material } G)$$

$$= -\int_G \phi \, dV \Big|_{t=-\infty}^{t=\infty} = 0 \quad \square$$

Therefore

$$\frac{\partial}{\partial t} \int_V \rho_g X_g \underline{v}\, dV + \left[\int_A X_g \rho_g \underline{v}\,\underline{v}\cdot\underline{i}\, dS \right]_x^{x+\delta x}$$

divergence theorem, $\underline{v}\cdot\underline{n} = 0$ on walls – or $X_g = 0$ at walls – bubbly flow

$$= \left[\int_A X_g \underset{\sim}{\sigma}_g \cdot \underline{i}\, dS \right]_x^{x+\delta x} - \int_V \underset{\sim}{\sigma}_g \cdot \underline{\nabla} X_g \, dV - \int_V \rho_g g X_g \underline{i}\, dV$$

divergence theor. $X_g = 0$ at walls [if $X_g \neq 0$ at walls then a wall stress – this applies for liquid]

Take the x component by dotting with $\underline{i}$: $v = \underline{v}\cdot\underline{i}$: & let $\delta x \to 0$

$$\frac{\partial}{\partial t} \int_A \rho_g X_g v \, dS + \frac{\partial}{\partial x} \left[\int_A X_g \rho_g v^2 \, dS \right]$$

$$= \frac{\partial}{\partial x} \int_A \underbrace{X_g \underline{i} \cdot \underset{\sim}{\sigma}_g \cdot \underline{i}}_{\underline{i}\cdot\underset{\sim}{\sigma}_g\cdot\underline{i} = \sigma_{g,11}} \, dS - \int_A \underset{\sim}{\sigma}_g \cdot \underline{\nabla} X_g \cdot \underline{i}\, dS - \int_A \rho_g g X_g \, dS$$

To be specific (question doesn't really say)

assume gas is inviscid so $\underset{\sim}{\sigma}_g = -p_g \underset{\sim}{I}$ $\underset{\sim}{I}$ = unit tensor

$$\Rightarrow \quad \underset{\sim}{i} \cdot \underset{\sim}{\sigma}_g \cdot \underset{\sim}{i} = -p_g$$

$$\& \quad (\underset{\sim}{\sigma}_g \cdot \underline{\nabla} X_g)_i = -p_g \delta_{ij} \frac{\partial X_g}{\partial x_j} = -p_g \frac{\partial X_g}{\partial x_i}$$

so $\underset{\sim}{\sigma}_g \cdot \underline{\nabla} X_g = -p_g \underline{\nabla} X_g$ and $\int_A X_g \, \underset{\sim}{i} \cdot \underset{\sim}{\sigma}_g \cdot \underset{\sim}{i} \, dS = -\int_A X_g \, p_g \, dS$

Define averages

$$\alpha = \frac{1}{A} \int_A X_g \, dS$$

$$\alpha \bar{\rho}_g = \frac{1}{A} \int_A \rho_g X_g \, dS \qquad \Rightarrow \bar{\rho}_g = \frac{\int \rho_g X_g \, dS}{\int X_g \, dS}$$

$$\alpha \bar{\rho}_g \bar{v} = \frac{1}{A} \int_A \rho_g X_g v \, dS \qquad \bar{v} = \frac{\int \rho_g X_g v \, dS}{\int \rho_g X_g \, dS}$$

$$\alpha \bar{\rho}_g \overline{v^2} = \frac{1}{A} \int_A \rho_g X_g v^2 \, dS \qquad \overline{v^2} = \frac{\int_A \rho_g X_g v^2 \, dS}{\int \rho_g X_g \, dS}$$

and

$$\alpha \bar{p}_g = \frac{1}{A} \int X_g p_g \, dS \qquad \bar{p}_g = \frac{\int X_g p_g \, dS}{\int X_g \, dS}$$

then

$$\frac{\partial}{\partial t} (\alpha \bar{\rho}_g \bar{v}) + \frac{\partial}{\partial x} (\alpha \bar{\rho}_g \overline{v^2}) = -\frac{\partial}{\partial x} (\alpha \bar{p}_g) + \frac{1}{A} \int_A p_g \underline{\nabla} X_g \cdot \underline{i} \, dS - \alpha \bar{\rho}_g g$$

as requested with

$$\frac{M_g}{A} = -\alpha \bar{\rho}_g g + \frac{1}{A} \int_A p_g \underline{\nabla} X_g \cdot \underline{i} \, dS$$

gas mass : $(\alpha \bar{\rho}_g)_t + (\alpha \bar{\rho}_g \bar{v})_x = 0$

liquid $[\rho_\ell (1-\alpha)]_t + [\rho_\ell (1-\alpha)\bar{u}]_x = 0$

(ρ_ℓ constant)

$$\frac{\partial}{\partial t}[\rho_\ell (1-\alpha)\bar{u}]_t + \frac{\partial}{\partial x}[\rho_\ell (1-\alpha)\bar{u^2}]_x$$

$$= -\frac{\partial}{\partial x}\underbrace{[(1-\alpha)\bar{p}_\ell]}_{\text{ignore longitudinal stress terms}} \cancel{+ \frac{1}{A}\int_A \rho_\ell \nabla \chi_g \cdot \underline{i}\, dS}$$

$$\cancel{+ \frac{1}{A}\int_A \chi}$$

$$\underbrace{-\frac{1}{A}\int_{\partial A} \tau_{\ell w}\, ds}_{\substack{\text{wall shear stress} \\ (\chi_\ell = 1 \text{ at wall})}} - \rho_\ell (1-\alpha) g + \frac{1}{A}\int_A \underset{\sim}{\sigma_\ell} \cdot \underset{\sim}{\nabla}\chi_g \cdot \underline{i}\, dS$$

$[-\frac{1}{A}\int_A \sigma_\ell \nabla \chi_\ell \cdot \underline{i}\, dS]$ $\chi_g = 1 - \chi_\ell$

of similar form & $\frac{M_\ell}{A} = -\rho_\ell (1-\alpha) g - \underbrace{\frac{1}{A}\int_{\partial A} \tau_{\ell w}\, ds}_{\text{wall stress}} - \frac{1}{A}\int_A p_g \underline{\nabla}\chi_g \cdot \underline{i}\, dS$

since $\underset{\sim}{\sigma_\ell} = \underset{\sim}{\sigma_g}$ at interface

(b) Now with the various assumptions,

$$\frac{M_g'}{A} = \int_A p_g \underline{\nabla}\chi_g \cdot \underline{i}\, dS - \alpha \bar{\rho}_g g - p \frac{\partial \alpha}{\partial x}$$

$$= -\alpha \bar{\rho}_g g + \frac{1}{A}\int_A (p_g - p) \underline{\nabla}\chi_g \cdot \underline{i}\, dS$$

& (neglecting wall stress) $\frac{M_\ell'}{A} = p\alpha_x - \rho_\ell (1-\alpha) g - \frac{1}{A}\int p_g \underline{\nabla}\chi_g \cdot \underline{i}\, dS$

$$= -\rho_\ell (1-\alpha) g - \frac{1}{A}\int_A (p_g - p) \underline{\nabla}\chi_g \cdot \underline{i}\, dS$$

with these & with $u=0$ & $\overline{v^2} = \bar{v}^2 = v^2$

$$(\alpha \rho_g)_t + (\alpha \rho_g v)_x = 0$$

$$\frac{\partial}{\partial t}(\alpha \rho_g v) + \frac{\partial}{\partial x}(\alpha \rho_g v^2)$$

$$= \alpha \rho_g (v_t + v v_x) = -(\alpha p)_x + p \alpha_x \overbrace{- \alpha \rho_g g \underbrace{- \frac{1}{A}\int_A (p - p_g)\nabla X_g \cdot \underline{i}\, dS}_{D}}^{M_g/A'}$$

The term $D = \frac{1}{A}\int_A (p-p_g)\nabla X_g \cdot \underline{i}\, dS$ is the drag on the bubbles

drag on a single bubble of radius a is $C_D \pi a^2 \rho_\ell |v| v$ whereas AD = drag force / unit length of tube

If there are n bubbles in a length δx of radius a

then $\alpha = \dfrac{n \frac{4}{3}\pi a^3}{A \delta x}$

& ~~ADx~~ $A D \delta x$ = total drag $= n C_D \pi a^2 \rho_\ell |v| v$

$$\Rightarrow D = \frac{\alpha A \delta x}{\frac{4}{3}\pi a^3} \frac{C_D \pi a^2 \rho_\ell |v| v}{A \delta x}$$

$$= \frac{3}{4a} \alpha C_D \rho_\ell |v| v$$

$$\Rightarrow \alpha \rho_g (v_t + v v_x) = -\alpha p_x - \alpha \rho_g g - \frac{3}{4a}\alpha C_D \rho_\ell |v| v$$

each bubble conserves mass so $\rho_g a^3$ is constant $\Rightarrow \underline{a = a_0 \left(\frac{\rho_0}{\rho_g}\right)^{1/3}}$

w.r.t. reference values a_0, ρ_0

Corresponding expression for

$$\frac{M_\ell'}{A} = -\rho_\ell(1-\alpha)g + \frac{3}{4a}\alpha c_D \rho_\ell |v|v$$

(e) Note $u=0$ so liquid mass ~~hydrostatic~~ $\Rightarrow \alpha_t = 0$

& $$0 = -\frac{\partial}{\partial x}\left[(1-\alpha)p\right] - p\alpha_x - \rho_\ell(1-\alpha)g + \frac{3}{4a}\alpha c_D \rho_\ell |v|v$$

$$= -(1-\alpha)p_x - \rho_\ell(1-\alpha)g + \frac{3}{4a}\alpha c_D \rho_\ell |v|v$$

Non-d gas equations

$x \sim L \quad p - p_0 \sim \rho_\ell g L \quad v \sim U = \sqrt{\dfrac{4a_0 g}{3c_D}} \quad \rho_g \sim \rho_0$

(write $\rho_g = \rho_0\rho$)

$t \sim \dfrac{L}{U}$

$$\Rightarrow (\alpha\rho)_t + (\alpha\rho v)_x = 0$$

$$\rho_0\frac{U^2}{L}\alpha\rho(v_t + vv_z) = \rho_\ell g \cdot -\alpha p_x - \alpha\rho_0 g\rho - \frac{3}{4a}c_D\rho_\ell U^2\alpha|v|v$$

$\therefore$ ~~ρ_0~~ $\rho_\ell g\alpha$

$$\Rightarrow \frac{\rho_0}{\rho_\ell}\frac{U^2}{gL}\rho(v_t + vv_z) = -p_x - \frac{\rho_0}{\rho_\ell}\rho - \frac{3c_D U^2}{4ag}|v|v$$

& $$\frac{3c_D U^2}{4ag} = \frac{3c_D}{4ag}\cdot\frac{4a_0 g}{3c_D} = \frac{a_0}{a} = \left(\frac{\rho_g}{\rho_0}\right)^{1/3} = \rho^{1/3}$$

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Note also $\dfrac{U^2}{gL} = \dfrac{4a_0 g}{3c_D gL} = \dfrac{4}{3c_D}\dfrac{a_0}{L}$

& $a_0 \ll L$ $\quad c_D \sim 1$ $\quad$ so $\dfrac{U^2}{gL} \ll 1$ & $\dfrac{\rho_0}{\rho_\ell} \ll 1$

$\Rightarrow \quad 0 \simeq -p_x \quad - \rho^{1/3}|v|v$

or back in dimensional terms

liquid $\quad 0 = -(1-\alpha)p_x - \rho_\ell(1-\alpha)g \quad + \dfrac{3}{4a}\alpha c_D \rho_\ell |v|v$

(all terms same size)

gas $\quad 0 \simeq \quad -\alpha p_x \qquad -\dfrac{3}{4a}\alpha c_D \rho_\ell |v|v$

add $\quad p_x = -\rho_\ell g(1-\alpha) \overset{(gas)}{=} -\dfrac{3}{4a} c_D \rho_\ell |v|v$

(d) ~~liquid mass~~ gas mass $\quad \alpha\rho_g v = b$ constant $\quad$ (>0 say)

so $\quad \rho_\ell g(1-\alpha) = \dfrac{3c_D}{4a_0}\rho_\ell v^2\left(\dfrac{\rho_g}{\rho_0}\right)^{1/3}$

$\simeq$

back to $\rho = \dfrac{\rho_g}{\rho_0}$ $\quad \Rightarrow \quad 1-\alpha = \dfrac{3c_D}{4a_0 g} v^2 \rho^{1/3}$

& $\alpha\rho v = \dfrac{b}{\rho_0}$

so
$$\alpha^2(1-\alpha) = \frac{3c_D}{4a_0 g}\,\alpha^2 v^2 \rho^{1/3}$$
$$= \frac{3c_D}{4a_0 g}\frac{b^2}{\rho_0^2}\,\rho^{-5/3}$$
$$\Rightarrow \rho = \left(\frac{3c_D b^2}{4a_0 g \rho_0^2}\right)^{\cancel{5/3}\,3/5} \frac{1}{[\alpha^2(1-\alpha)]^{3/5}}$$

of requested form.

~~I~~ If $p = c^2\rho_g = c^2\rho_0\rho$

then
$$p_x = c^2\rho_0\rho_x = -\rho_\ell g(1-\alpha)$$
$$\Rightarrow c^2\rho_0\left(\frac{3c_D b^2}{4a_0 g\rho_0^2}\right)^{3/5}\frac{3}{5}\,\frac{-1}{[\alpha^2(1-\alpha)]^{8/5}}\cdot(2\alpha-3\alpha^2)\alpha_x = -\rho_\ell g(1-\alpha)$$
$$\Rightarrow \alpha_x = \frac{\rho_\ell g}{c^2\rho_0}\,\frac{5}{3}\left(\frac{4a_0 g\rho_0^2}{3c_D b^2}\right)^{3/5}\alpha^{\frac{16}{5}}(1-\alpha)^{13/5}$$