Topics in fluid mechanics

PROBLEM SHEET 2.

1. An incompressible two-dimensional flow from a slit of width d falls vertically under gravity. Define *vertical* and *horizontal* coordinates x and z, with corresponding velocity components u and w. The stream is symmetric with free interfaces at $z = \pm s$, on which no stress conditions apply. Write down the equations and boundary conditions in terms of the deviatoric stress components $\tau_1 = \tau_{11} = -\tau_{33}$ and $\tau_3 = \tau_{13} = \tau_{31}$, and by scaling lengths with l, velocities with the inlet velocity U, and choosing suitable scales for time t and the pressure and stresses, show that the equations take the form

$$u_x + w_z = 0,$$

$$Re\,\dot{u} = -p_x + \tau_{1x} + \tau_{3z} + 1,$$

$$Re\,\dot{w} = -p_z + \tau_{3x} - \tau_{1z},$$

where you should define $\dot{u}$, the Reynolds number Re, and write down expressions for τ_1 and τ_3.

Now define $\varepsilon = \dfrac{d}{l}$, and assume it is small. Find a suitable rescaling of the equations, and show that the vertical momentum equation takes the approximate form

$$h[Re\,\dot{u} - 1] = 4(hu_x)_x,$$

where $u = u(x,t)$ and h is the stream width.

Show also that

$$h_t + (hu)_x = 0.$$

Explain why suitable boundary conditions are

$$h = u = 1 \quad \text{at} \quad x = 0, \qquad hu_x \to 0 \quad \text{as} \quad x \to \infty.$$

Write down a single second order equation for u in steady flow. If $Re = 0$, find the solution.

If $Re > 0$, find a pair of first order equations for $v = \ln u$ and $w = v_x$. (*Note: w here is no longer the horizontal velocity.*) Show that $(\infty, 0)$ is a saddle point, and that a unique solution satisfying the boundary conditions exists. If $Re \gg 1$ (but still $\varepsilon^2 Re \ll 1$), show (by rescaling $w = W/Re$ and $x = Re\,X$) that the required trajectory hugs the W–nullcline, and thus show that in this case

$$u \approx \left(1 + \frac{2x}{Re}\right)^{1/2}.$$

2. Saturated groundwater flows between an impermeable basement $z = b(\mathbf{x})$ and a phreatic surface $z = s(\mathbf{x}, t)$, on which the pressure is atmospheric (p_a) and the kinematic condition takes the form

$$w = \phi s_t + \mathbf{u}.\boldsymbol{\nabla} s - r_p;$$

here $\mathbf{x} = (x, y)$ is the horizontal space coordinate and $\mathbf{u}$ the horizontal flux: $\boldsymbol{\nabla}$ is the horizontal gradient, w the vertical flux, and r_p is the surface source due to rainfall (volume per unit area per unit time). Write down the equations describing mass conservation and Darcy's law, and also the boundary condition of no normal flow at the lower boundary. By integrating the mass conservation equation, show that

$$\phi h_t + \boldsymbol{\nabla}.\mathbf{q} = r_p, \quad \mathbf{q} = \int_b^s \mathbf{u}\, dz, \qquad (*)$$

where $h = s - b$ is the depth of the flow.

Use the velocity scale $K = \dfrac{k\rho g}{\mu}$ to scale $(\mathbf{u}, w)$, the (unknown) depth scale d to scale lengths $(\mathbf{x}, z)$, scale $p - p_a$ with $\rho g d$, and write down the resultant forms for the mass conservation equation and Darcy's law.

Now suppose that a more appropriate horizontal length scale is l, and that $\varepsilon = \dfrac{d}{l} \ll 1$. Show that an appropriate re-scaling of the equations is $\mathbf{x} \sim \dfrac{1}{\varepsilon}$, $\mathbf{u} \sim \varepsilon$, $w \sim \varepsilon^2$, and write down the re-scaled forms of the equations. Hence provide an approximate solution for p and $\mathbf{u}$, and deduce that equation $(*)$ takes the dimensionless form

$$h_t = \boldsymbol{\nabla}.[h\boldsymbol{\nabla} s] + 1,$$

providing the scale for t is $t \sim \dfrac{\phi d}{\varepsilon^2 K}$, and that d is chosen as

$$d = l\left(\frac{r}{K}\right)^{1/2}.$$

Deduce that the assumption of shallow water table slope is valid if $r \ll K$. What happens if $r > K$?

Find the solution for h in steady flow due to precipitation on a circular island with base $b = 0$ and $h = 0$ at (radius) $r = 1$.

3. Write down conservation laws for liquid and solid mass in a one-dimensional consolidating soil in $0 < z < h(t)$, where ϕ is the porosity, v is the liquid velocity, and w is the solid velocity. Also write down a relation for Darcy's law. Give appropriate boundary conditions for the flow, assuming the base is impermeable and the surface is open to the atmosphere.

Assuming that the overburden pressure P is hydrostatic, i. e.,

$$\frac{\partial P}{\partial z} = -[\rho_s(1-\phi) + \rho\phi]g,$$

and that the effective pressure is

$$P - p = p_e(\phi),$$

show that the model can be reduced to the equation

$$\phi_t + V_z = (D\phi_z)_z\,,$$

and give expressions for the functions $V(\phi)$ and $D(\phi)$. Write down the boundary conditions for ϕ on $z = 0$ and $z = h$, and the kinematic condition to determine h. Hence determine the steady state solution implicitly in terms of an integral with respect to ϕ. If now an additional load ΔP is applied at the surface, find the corresponding change in surface porosity $\Delta\phi$, and show that the settlement is

$$\Delta h = \frac{\Delta P}{\Delta\rho(1-\phi_0)g}, \quad \Delta\rho = \rho_s - \rho.$$

For the particular case where V and D are constant, show that small perturbations to the steady state $\phi^{(0)}, h_0$ of the form

$$\phi = \phi^{(0)} + \Phi, \quad h = h_0 + \eta,$$

satisfy the linear system

$$\Phi_t = D\Phi_{zz},$$

$$\Phi_t + \frac{V}{1-\phi_0}\Phi_z = 0 \quad \text{on} \quad z = h_0,$$

$$\Phi_z = 0 \quad \text{on} \quad z = 0,$$

and deduce that normal modes of vertical wavenumber k decay exponentially at a rate $-Dk^2$, and that

$$\tan kh_0 = -\frac{kh_0}{Pe}, \quad Pe = \frac{Vh_0}{(1-\phi_0)D}.$$

Hence show that the least stable decay rate is $\pi^2 D/h_0^2$ for $Pe \gg 1$. What is it if $Pe \ll 1$?

4. A two-dimensional, incompressible fluid flow has velocity $\mathbf{u} = (u, 0, w)$, and depends only on the coordinates x and z. Show that there is a stream function ψ satisfying $u = -\psi_z$, $w = \psi_x$, and that the vorticity

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u} = -\nabla^2\psi\mathbf{j},$$

and thus that

$$\mathbf{u} \times \boldsymbol{\omega} = (\psi_x\nabla^2\psi, 0, \psi_z\nabla^2\psi),$$

and hence

$$\boldsymbol{\nabla} \times (\mathbf{u} \times \boldsymbol{\omega}) = (\psi_x\nabla^2\psi_z - \psi_z\nabla^2\psi_x)\mathbf{j}.$$

Use the vector identity $(\mathbf{u}\,.\boldsymbol{\nabla})\mathbf{u} = \boldsymbol{\nabla}(\frac{1}{2}u^2) - \mathbf{u} \times \boldsymbol{\omega}$ to show that

$$\boldsymbol{\nabla} \times \frac{d\mathbf{u}}{dt} = \left[-\nabla^2\psi_t - \psi_x\nabla^2\psi_z + \psi_z\nabla^2\psi_x\right]\mathbf{j}.$$

Show also that

$$\boldsymbol{\nabla} \times \theta\mathbf{k} = -\theta_x\mathbf{j},$$

and use the Cartesian identity

$$\nabla^2 \equiv \mathrm{grad}\,\mathrm{div} - \mathrm{curl}\,\mathrm{curl}$$

to show that

$$\boldsymbol{\nabla} \times \nabla^2\mathbf{u} = -\nabla^4\psi\,\mathbf{j},$$

and hence deduce that the dimensionless momentum equation for Rayleigh–Bénard convection, assuming the Boussinesq approximation, can be written in the form

$$\frac{1}{Pr}\left[\nabla^2\psi_t + \psi_x\nabla^2\psi_z - \psi_z\nabla^2\psi_x\right] = Ra\,\theta_x + \nabla^4\psi.$$