Topics in fluid mechanics

PROBLEM SHEET 3.

1. The Boussinesq equations of two-dimensional thermal convection can be written in the dimensionless form

$$\begin{aligned} \boldsymbol{\nabla}.\mathbf{u} &= 0, \\ \frac{1}{Pr}[\mathbf{u}_t + (\mathbf{u}\,.\boldsymbol{\nabla})\mathbf{u}] &= -\boldsymbol{\nabla} p + \nabla^2\mathbf{u} + Ra\,T\hat{\mathbf{k}}, \\ T_t + \mathbf{u}.\boldsymbol{\nabla} T &= \nabla^2 T. \end{aligned}$$

Explain the meaning of these equations, and write down appropriate boundary conditions assuming stress-free boundaries.

By introducing a suitably defined stream function, show that these equations can be written in the form

$$\begin{aligned} \frac{1}{Pr}\left[\nabla^2\psi_t + \psi_x\nabla^2\psi_z - \psi_z\nabla^2\psi_x\right] &= Ra\,T_x + \nabla^4\psi, \\ T_t + \psi_x T_z - \psi_z T_x &= \nabla^2 T, \end{aligned}$$

with the associated boundary conditions

$$\begin{aligned} \psi = \nabla^2\psi = 0 \quad &\text{at} \quad z = 0, 1, \\ T = 0 \quad &\text{at} \quad z = 1, \\ T = 1 \quad &\text{at} \quad z = 0, \end{aligned}$$

and write down the conductive steady state solution.

By linearising about this steady state, show that

$$\frac{1}{Pr}\left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2\psi_t = \left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^4\psi + Ra\,\psi_{xx},$$

and deduce that solutions are $\psi = e^{\sigma t}\sin kx \sin m\pi z$, and thus that

$$(\sigma + K^2)\left(\frac{\sigma}{K^2 Pr} + 1\right) - \frac{Ra\,k^2}{K^4} = 0, \quad K^2 = k^2 + m^2\pi^2.$$

By considering the graph of this expression as a function of σ, show that oscillatory instabilities can not occur, and hence derive the critical Rayleigh number for the onset of convection.

2. The scaled Boussinesq equations for two-dimensional thermal convection at infinite Prandtl number and large Rayleigh number R in $0 < x < a$, $0 < z < 1$, can be written in the form

$$\begin{aligned} \omega &= -\nabla^2\psi, \\ \nabla^2\omega &= \frac{1}{\delta}T_x, \\ \psi_x T_z - \psi_z T_x &= \delta^2\nabla^2 T, \end{aligned}$$

where $\delta = R^{-1/3}$. Suitable boundary conditions for these equations which represent convection in a box with stress free boundaries, as appropriate to convection in the Earth's mantle, are given by

$$\psi = \omega = 0 \quad \text{on} \quad x = 0,\ a, \quad z = 0, 1,$$

$$T = \tfrac{1}{2} \quad \text{on} \quad z = 0, \quad T = -\tfrac{1}{2} \quad \text{on} \quad z = 1, \quad T_x = 0 \quad \text{on} \quad x = 0, a.$$

Show that, if $\delta \ll 1$, there is an interior 'core' in which $T \approx 0$, $\nabla^4\psi = 0$.

By writing $1-z = \delta Z$, $\psi = \delta\Psi$ and $\omega = \delta\Omega$, show that $\Psi \approx u_s(x)Z$, and deduce that the temperature in the thermal boundary layer at the surface is described by the approximate equation

$$u_s T_x - Z u_s' T_Z \approx T_{ZZ},$$

with

$$T = -\tfrac{1}{2} \quad \text{on} \quad Z = 0, \quad T \to 0 \quad \text{as} \quad Z \to \infty.$$

If u_s is constant, find a similarity solution, and show that the scaled surface heat flux $q = \partial T/\partial Z|_{Z=0}$ is given by

$$q = \frac{1}{2}\sqrt{\frac{u_s}{\pi x}}.$$

Can this form of solution be extended to the case where $u_s(x)$ is not constant?

3. Write down a dimensional set of equations to describe double-diffusive convection in a layer of fluid subjected to precribed positive temperature and concentration differences ΔT and Δc across the layer.

Suppose that $\rho = \rho_0[1-\alpha(T-T_0)+\beta c]$. Non-dimensionalise the model, and show how the Boussinesq approximation leads to the dimensionless set of equations

$$\begin{aligned} \boldsymbol{\nabla}.\mathbf{u} &= 0, \\ \frac{1}{Pr}[\mathbf{u}_t + (\mathbf{u}\,.\boldsymbol{\nabla})\mathbf{u}] &= -\boldsymbol{\nabla}p + \nabla^2\mathbf{u} + Ra\,T\hat{\mathbf{k}} - Rs\,c\hat{\mathbf{k}}, \\ T_t + \mathbf{u}.\boldsymbol{\nabla}T &= \nabla^2 T, \\ c_t + \mathbf{u}\,.\boldsymbol{\nabla}c &= \frac{1}{Le}\nabla^2 c. \end{aligned}$$

Give the definitions of the dimensionless parameters Ra, Rs, Le and Pr.

By seeking solutions of the equations, linearised about a suitable steady state, proportional to $\exp(ikx + \sigma t)$, show that σ satisfies a cubic of the form

$$p(\sigma) = \sigma^3 + a\sigma^2 + b\sigma + c = 0,$$

and give the definitions of a, b, c.

Suppose that a, b and c are positive. Suppose, firstly, that the roots are all real. Show in this case that they are all negative.

Now suppose that one root $(-\alpha)$ is real and the other two are complex conjugates $\beta \pm i\gamma$. Show that $\alpha > 0$. Show also that $\beta < 0$ if $a > \alpha$. Show that $a > \alpha$ if $p(-a) < 0$, and hence show that $\beta < 0$ if $c < ab$.

Show that $a, b, c > 0$ if $Ra < 0$, $Rs > 0$, and show that also $c < ab$.

Deduce that direct instability ($\operatorname{Im}\sigma = 0$) occurs if

$$Ra > Le\, Rs + R_c,$$

where you should define R_c.

Show that oscillatory instability ($\operatorname{Im}\sigma \neq 0$) occurs if

$$Ra > \frac{\left(Pr + \dfrac{1}{Le}\right) Rs}{1 + Pr} + \frac{\left(Pr + \dfrac{1}{Le}\right)\left(1 + \dfrac{1}{Le}\right) R_c}{Pr}.$$

4. An isolated turbulent cylindrical plume in a stratified medium of density $\rho_0(z)$ is described by the inviscid Boussinesq equations

$$\rho(uu_r + wu_z) = -p_r,$$

$$\rho(uw_r + ww_z) = -p_z - \rho g,$$

$$u\rho_r + w\rho_z = 0,$$

$$\frac{1}{r}(ru)_r + w_z = 0,$$

where (r, z) are cylindrical coordinates, (u, w) the corresponding velocity components, p the pressure, ρ the density, ρ_0 the reference density, and g is the acceleration due to gravity. If $\rho = \rho_0 - \Delta\rho$, explain what is meant by the Boussinesq approximation.

Suppose the edge of the plume is at radius $r = b$, such that $w = 0$ there. Suppose also that the plume entrains ambient fluid, such that

$$(ru)|_b = -b\alpha\bar{w},$$

where $\bar{w}$ denotes the cross-sectional average value of w. Deduce that the plume volume flux

$$Q = 2\pi \int_0^b rw\,dr$$

satisfies

$$\frac{dQ}{dz} = 2\pi\alpha b\bar{w}.$$

The momentum flux is defined by

$$M = 2\pi \int_0^b rw^2\,dr.$$

Show, assuming that

$$\frac{\partial p}{\partial z} = -\rho_0 g$$

throughout the plume, that

$$\frac{dM}{dz} = 2\pi \int_0^b rg'\,dr,$$

where

$$g' = \frac{g\Delta\rho}{\rho_0}.$$

Why would the hydrostatic approximation be appropriate?

The buoyancy flux is defined by

$$B = 2\pi \int_0^b rwg'\,dr;$$

assuming $g' = 0$ at $r = b$, show that

$$\frac{dB}{dz} \approx -N^2 Q,$$

where the Brunt–Väisälä frequency N is defined by

$$N = \left(-\frac{g\rho_0'(z)}{\rho_0}\right)^{1/2},$$

and it is assumed that $\Delta\rho \ll \rho_0$.