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1. Introduction

- Perturbation methods exploit a small, or large, parameter to make systematic, precise approximations.

Difficult to give rules for perturbation methods, only guidelines.

- Hinch, Bender & Orszag, Supplementary Notes online.

2. Algebraic Equations

Example

$$x^2 + \varepsilon x - 1 = 0, \quad |\varepsilon| \ll 1.$$

$$x = \left[-\frac{\varepsilon}{2} \pm \sqrt{1 + \left(\frac{\varepsilon}{2}\right)^2} \right] \stackrel{\text{binomial expansion}}{=} \begin{cases} 1 - \varepsilon/2 + \varepsilon^2/8 + \dots \\ -1 - \varepsilon/2 - \varepsilon^2/8 + \dots \end{cases}$$

for $|\varepsilon/2| < 1$.
for convergence

In addition to convergence, truncated expansions give good approximations to the roots when $|\varepsilon| \ll 1$.

For $\varepsilon = 0.1$ and the positive root

$x \approx$	1	1st term	} exact root is
	0.95	2nd term	
	0.95125	3rd term	
	0.951249	4th term	
			0.95124922...

Solved, then approximated ... usually we approximate, then solve ②
2019.

2.1 Iterative method $x^2 + \epsilon x - 1 = 0$

For positive root $x = \sqrt{1 - \epsilon x}$ by rearrangement.

Consider iteration $x_{n+1} = g_\epsilon(x_n) := \sqrt{1 - \epsilon x_n}$

Note, if x^* is a root, so that $x^* = g_\epsilon(x^*)$ then if $|x_n - x^*|$ is small,

$$\begin{aligned} x_{n+1} - x^* &= g_\epsilon(x_n) - x^* = g_\epsilon(x^* + (x_n - x^*)) - x^* \\ &= \underbrace{(g_\epsilon(x^*) - x^*)}_0 + (x_n - x^*) g'_\epsilon(x^*) + \dots \end{aligned}$$

Also $g'_\epsilon(x^*) = \frac{-\epsilon/2}{\sqrt{1 - \epsilon x^*}} \approx -\epsilon/2 \therefore |x_{n+1} - x^*| \approx \left| \frac{\epsilon}{2} \right| |x_n - x^*|$

Hence iteration converges. at least if initial guess x_0 is sufficiently close to x^*

Beginning with $x_0 = 1$, $x_1 = \sqrt{1 - \epsilon} = \underbrace{1 - \epsilon/2 - \epsilon^2/8 - \epsilon^3/16 + \dots}_{\text{Correct}}$

$$\begin{aligned} x_2 &= \sqrt{1 - \epsilon \left(1 - \frac{\epsilon}{2} + \dots\right)} \\ &= \underbrace{1 - \frac{\epsilon}{2} \left(1 - \frac{\epsilon}{2} + \dots\right) - \frac{\epsilon^2}{8} \left(1 - \frac{\epsilon}{2} + \dots\right)^2 - \frac{\epsilon^3}{16} \left(1 - \frac{\epsilon}{2} + \dots\right)^3}_{\text{Correct}} + \underbrace{\dots}_{\text{Not correct}} \end{aligned}$$

At each iteration, more terms correct, but more work required. If solution not known, can only confirm terms are correct by performing a further iteration and checking they do not change.

For fast convergence, ideally want $g_\epsilon(x)$ such that $g'_\epsilon(x) \rightarrow 0$ as $\epsilon \rightarrow 0$. (5)

Note: choice of $g_\epsilon(x)$ not unique.

2.2 Expansion Method (Much more common)

$$x^2 + \epsilon x - 1 = 0$$

For $\epsilon = 0$, $x = \pm 1$.

to be determined
no dependence on ϵ

Positive root let $x = 1 + \epsilon x_1 + \epsilon^2 x_2 + \dots$

No dependence critical.

$$(1 + \epsilon x_1 + \epsilon^2 x_2 + \dots)^2 + \epsilon(1 + \epsilon x_1 + \epsilon^2 x_2 + \dots) - 1 = 0$$

Term involving ϵ^0

$$1 - 1 = 0$$

$$\epsilon^1 \quad 2x_1 + 1 = 0 \quad \therefore x_1 = -1/2$$

$$\epsilon^2 \quad 2x_2 + x_1^2 + x_1 = 0 \quad \therefore x_2 = 1/8 \text{ etc.}$$

$$\therefore x = 1 - 1/2 \epsilon + 1/8 \epsilon^2 + \dots$$

set coefficients of powers of ϵ to zero for each power... as this is to hold for all sufficiently small ϵ .

Caveat Must know/assume form of expansion

2.3 Singular Perturbations

$$\epsilon x^2 + x - 1 = 0 \quad |\epsilon| \ll 1$$

$\epsilon = 0$, one root $x = 1$. $\epsilon \neq 0$ two roots.

Singular ... the case with $\epsilon = 0$ differs in an important way from the case with $\epsilon \rightarrow 0$.

Non-singular problems are regular.

Solve

$$x = \frac{1}{2\varepsilon} \left[-1 \pm \sqrt{1 + 4\varepsilon} \right]$$

$$= \begin{cases} 1 - \varepsilon + 2\varepsilon^2 + \dots \\ -\frac{1}{2} - 1 + \varepsilon - 2\varepsilon^2 + \dots \end{cases}$$

for $|4\varepsilon| < 1$
by binomial expansion

Second root blows up as $\varepsilon \rightarrow 0$.

Both tend to zero for $x \approx x_0$, the respective root, and $\varepsilon \rightarrow 0$.

Iterative method

Note $g_\varepsilon'(x) = -2\varepsilon x \approx -2\varepsilon$ for 1st case

$g_\varepsilon'(x) = -\frac{1}{2} \cdot \frac{1}{2} x_0 \approx -\varepsilon$ for 2nd case

$g_\varepsilon(x) = 1 - \varepsilon x^2$ for 1st root

$g_\varepsilon(x) = \frac{1-x}{\varepsilon x}$ for 2nd root

Both derivatives are small near their respective roots and ~~are~~ tend to zero as $\varepsilon \rightarrow 0$.

Expansion method (2nd root)

2nd root differs from above example

Let $x = \frac{x_{-1}}{\varepsilon} + x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$ and consider $\varepsilon x^2 + x - 1 = 0$

At ε^{-1}

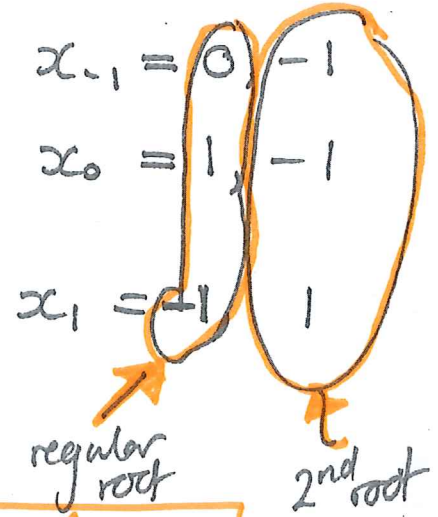
$x_{-1}^2 + x_{-1} = 0$

At ε^0

$2x_{-1}x_0 + x_0 - 1 = 0$

At ε^1

$(2x_{-1}x_1 + x_0^2) + x_1 = 0$



To be more systematic.

∴ If we have found regular root do not explore case $x_{-1} = 0$

Rescaling

Zero for x_{-1}

let $x = X/\varepsilon \Rightarrow X^2 + X - \varepsilon = 0$, regular.

Finding correct starting point for expansion same as finding a rescaling that makes problem regular.
rescaling that makes problem regular

2.4 Finding the correct rescaling

(5)

Systematic approach

$$x = \delta(\varepsilon)X \quad \text{with } X \text{ strictly order 1 as } \varepsilon \rightarrow 0$$

We have

$$\varepsilon \delta^2 X^2 + \delta X - 1 = 0$$

Vary δ from very small to large to identify dominant balances, where at least 2 terms are of the same order of magnitude, with all other terms smaller.

Scalings that yield dominant balances are distinguished limits.

$\delta \ll 1$	$\textcircled{1} \ll \textcircled{2} \ll \textcircled{3}$	No balance
$\delta = 1$	$\textcircled{1} \ll \textcircled{2} \sim \textcircled{3}$ <div style="border: 1px solid orange; padding: 2px; display: inline-block; margin-left: 20px;"> same order of magnitude </div>	Balance, regular root $x = 1 + \text{small}$ $X = 1 + \text{small}$
$1 \ll \delta \ll \frac{1}{\varepsilon}$	$\textcircled{1} \ll \textcircled{2} \gg \textcircled{3}$	No balance
$\delta = \frac{1}{\varepsilon}$	$\textcircled{1} \sim \textcircled{2} \gg \textcircled{3}$	Balance, singular root $x = \frac{1}{\varepsilon} + \dots$ $X = -1 + \text{small}$
$\delta \gg \frac{1}{\varepsilon}$	$\textcircled{1} \gg \textcircled{2} \gg \textcircled{3}$	No balance

\therefore Distinguished limits are $\delta = 1, \frac{1}{\varepsilon}$.

Alternative approach: Pairwise comparison

- ① ~ ② $\epsilon \delta^2 \sim \delta$ i.e. $\delta \sim \frac{1}{\epsilon}$ and ① ~ ② \gg ③ \therefore Singular root.
- ① ~ ③ $\epsilon \delta^2 \sim 1$ i.e. $\delta \sim \frac{1}{\sqrt{\epsilon}}$ and ① ~ ③ \ll ② \therefore No dominant balance
- ② ~ ③ $\delta \sim 1$ ② ~ ③ \gg ① \therefore Regular root.

as ② dominates in isolation

2.5 Non-integer powers

$$(1-\epsilon)x^2 - 2x + 1 = 0 \quad |\epsilon| \ll 1$$

$$x = \frac{1 \pm \sqrt{\epsilon}}{1-\epsilon} = 1 \pm \sqrt{\epsilon} + \epsilon \pm \epsilon^{3/2} + \dots$$

double root... sign of danger

With $\epsilon = 0$ $(x-1)^2 = 0 \therefore x = 1.$

Try $x = 1 + \epsilon x_1 + \epsilon^2 x_2 + \dots$

Know this will go wrong
Objective is to see how it goes wrong so we know what to do if we see analogous behaviour in the expansion method when we do not know the solution

At ϵ^0 $1 - 2 + 1 = 0$

At ϵ^1 $-1 + 2x_1 - 2x_1 = 0$ No solution, "unless x_1 blows up in some sense."

Try $x = 1 + \epsilon^{1/2} x_{1/2} + \epsilon x_1 + \epsilon^{3/2} x_{3/2} + \dots$

At ϵ^0 $1 - 2 + 1 = 0$

At $\epsilon^{1/2}$ $2x_{1/2} - 2x_{1/2} = 0$

At ϵ $2x_1 + x_{1/2}^2 - 1 - 2x_1 = x_{1/2}^2 - 1 = 0$
 $\therefore x_{1/2} = \pm 1$

etc.

2.6 Finding the correct expansion sequence

(7)

Let $x = 1 + \delta_1(\epsilon)x_1$ where $\delta_1(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$,
 with x_1 strictly order one.
 root when $\epsilon=0$

$$(1-\epsilon)(1+\delta_1 x_1)^2 - 2(1+\delta_1 x_1) + 1 = 0$$

$$1 + 2\delta_1 x_1 + \delta_1^2 x_1^2 - \epsilon(1 + 2\delta_1 x_1 + \delta_1^2 x_1^2) - 2 - 2\delta_1 x_1 + 1 = 0$$

$$\delta_1^2 x_1^2 - \epsilon - 2\epsilon\delta_1 x_1 - \epsilon\delta_1^2 x_1^2 = 0 \quad (*)$$

①
②
③
④

Seek dominant balance ④ \ll ① always ③ \ll ② always

\therefore ④③ not in dominant balance \therefore ① \sim ② \therefore $\epsilon \sim \delta_1^2$

\therefore let $\delta_1 = \sqrt{\epsilon}$

$\therefore x = 1 + \epsilon^{1/2}x_1 + \delta_2(\epsilon)x_2 + \dots$ } From (*) we get at $O(\epsilon)$
 with $\delta_2(\epsilon) \ll \delta_1 = \epsilon^{1/2}$ } $x_1^2 - 1 = 0 \therefore$ $x_1 = \pm 1$

With root with $x_1 = 1$

$\therefore x = 1 + \epsilon^{1/2} + \delta_2(\epsilon)x_2 + \dots$

$$(1-\epsilon)(1 + \epsilon^{1/2} + \delta_2 x_2 + \dots)^2 - 2(1 + \epsilon^{1/2} + \delta_2 x_2 + \dots) + 1 = 0$$

(Lots of algebra.)
 \rightarrow retaining all terms

$$2\epsilon^{1/2}\delta_2 x_2 + \delta_2^2 x_2^2 - 2\epsilon^{3/2} - \epsilon^2 - 2\epsilon\delta_2 x_2 - 2\epsilon^{3/2}\delta_2 x_2 - \epsilon\delta_2^2 x_2^2 = 0$$

①
②
③
④
⑤
⑥
⑦

Want dominant terms, but only know $\delta_2 \ll \epsilon^{1/2}$
 $x_2 \sim$ order one

④ \ll ③ for terms with no δ_2 .

For terms with δ_2 ,

$$\textcircled{1} \gg \textcircled{7}, \quad \textcircled{1} \sim \varepsilon^{1/2} \delta_2 \quad \textcircled{7} \sim \varepsilon \delta_2^2 = (\varepsilon^{1/2} \delta_2)(\varepsilon^{1/2} \delta_2) \\ \sim \textcircled{1} \varepsilon^{1/2} \delta_2 \ll \textcircled{1}$$

Similarly for all other terms

\therefore Dominant balance is between ① and ③

$$2\varepsilon^{1/2} \delta_2 x_2 - 2\varepsilon^{3/2} = 0$$

$$\therefore \delta_2 x_2 = \varepsilon \quad \text{let } \delta_2 = \varepsilon, x_2 = 1$$

$$\therefore x = 1 + \varepsilon^{1/2} + \varepsilon + \dots$$

2.7 Iterative Method (again)

Useful when expansion form not known

$$(1-\varepsilon)x^2 - 2x + 1 = 0$$

$$(x-1)^2 = \varepsilon x^2$$

For root > 1

$$x_{n+1} = g_\varepsilon(x_n) := 1 + \varepsilon^{1/2} x_n$$

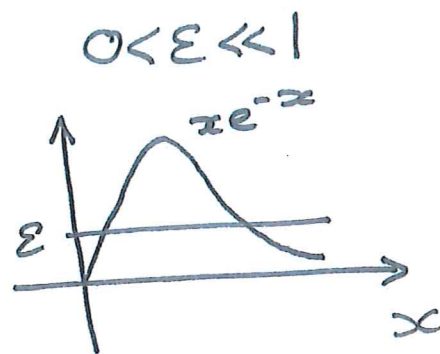
Note $g'_\varepsilon(x_n) \rightarrow 0$ as $\varepsilon \rightarrow 0$

$\therefore x_0 = 1$ solution if $\varepsilon = 0$
 $x_1 = 1 + \varepsilon^{1/2}, x_2 = \dots$ etc
 generates sequence

2.8 Logarithms

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Example $x e^{-x} = \varepsilon$



Root near $x=0$ easy to find.

[let $x = 0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$]

solution
if $\varepsilon=0$

Taylor expand $x \exp(-x)$ about zero ... generates integer powers of x and hence integer powers of $\varepsilon x_1, \varepsilon^2 x_2$, etc. Thus a series constructed from a sequence of integer powers of ε will enable a balance. Hence the form of the series.

Other root $\rightarrow \infty$ as $\varepsilon \rightarrow 0$

Hence the form of the expansion to find approximate solution is not immediately clear *for x large*

Take logs

$$\textcircled{1} x - \log \textcircled{2} x - \log \textcircled{3} \frac{1}{\varepsilon} = 0$$

For x large $|\textcircled{1}| \gg |\textcircled{2}| \therefore \textcircled{2}$ not in dominant balance
 $\therefore x \sim \log \frac{1}{\varepsilon}$ as $\varepsilon \rightarrow 0^+$

Suggests $x_{n+1} = g_\varepsilon(x_n) = \log(x_n) + \log\left(\frac{1}{\varepsilon}\right)$

Note $g'_\varepsilon(x) = \frac{1}{x}$

$g'_\varepsilon(x^*) \sim \frac{1}{\log\left(\frac{1}{\varepsilon}\right)} \rightarrow 0$ as $\varepsilon \rightarrow 0^+$ [but slow convergence]