4.2 Failure of Integration by Parts

Example $\quad I(x) = \int_0^\infty e^{-xt^2}\,dt = \frac{1}{2}\sqrt{\frac{\pi}{x}}$ for $x>0$.

let $u = x^{1/2} t$

Attempt (Parts)

$$I(x) = \int_0^\infty \left(\frac{-1}{2xt}\right)\left(-2xte^{-xt^2}\right)dt$$

$$= \underbrace{\left[\frac{e^{-xt^2}}{-2xt}\right]_0^\infty - \int_0^\infty \frac{e^{-xt^2}}{2xt^2}\,dt}_{\text{does not exist; fractional power in } x \text{ not picked up by this type of expansion}}$$

$\therefore$ Integration by parts simple but inflexible, of limited use.

Also does not work when dominant contribution to integral is from domain interior (need one limit to dominate).

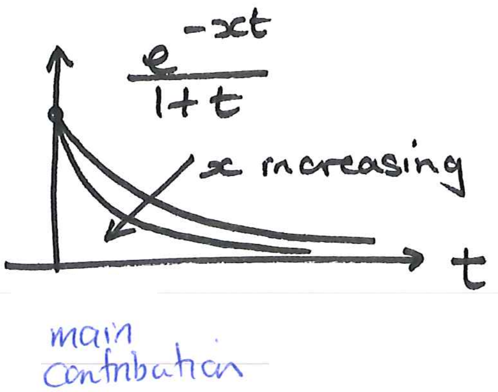
4.3 Laplace's Method

General technique for the asymptotic expansion as $x\to\infty$ of

$$I(x) = \int_a^b f(t)\,e^{x\varphi(t)}\,dt$$

with $[a,b] \subseteq \mathbb{R}$ and f, φ continuous real functions on $[a,b]$.

Example $\quad I(x) = \int_0^1 \frac{e^{-xt}}{1+t}\,dt$



$$I(x) = \overbrace{\int_0^{\varepsilon} \frac{e^{-xt}}{1+t}\,dt}^{I_1(x)} + \overbrace{\int_{\varepsilon}^{1} \frac{e^{-xt}}{1+t}\,dt}^{I_2(x)}$$

with $0 < 1/x \ll \varepsilon \ll 1.$

$$I_1(x) = \frac{1}{x}\int_0^{x\varepsilon} \frac{e^{-s}}{1+s/x}\,ds$$

$s/x < x\varepsilon/x = \varepsilon \ll 1$

$\therefore$ within radius of convergence and expansion uniform.

$$= \frac{1}{x}\int_0^{x\varepsilon} e^{-s}\left(\sum_{n=0}^{\infty}\left(\frac{-s}{x}\right)^n\right)ds$$

$$= \frac{1}{x}\sum_{n=0}^{\infty}\left[\int_0^{x\varepsilon} s^n e^{-s}\,ds\right]\frac{(-1)^n}{x^n}$$

$$\int_0^{x\epsilon} s^n e^{-s}\,ds = \int_0^{\infty} s^n e^{-s}\,ds - \int_{x\epsilon}^{\infty} e^{-s}s^n\,ds = n! - \underbrace{\int_{x\epsilon}^{\infty} s^n e^{-s}\,ds}_{K_n}$$

$$K_n \overset{\text{parts}}{=} \underbrace{(x\epsilon)^n e^{-x\epsilon}}_{\substack{\text{exponentially small}\\ \text{for fixed } n \text{ as } x\epsilon \gg 1.}} + n\int_{x\epsilon}^{\infty} s^{n-1}e^{-s}\,ds = \text{exponentially small} + nK_{n-1}$$

$$\therefore K_n = (n!)\int_0^{\infty} e^{-s}\,ds + \text{exponentially small} = (n!)e^{-x\epsilon} + \text{exponentially small} \ll n!$$

$$\therefore I_1 = \frac{1}{x}\sum_{n=0}^{\infty}\left(\int_0^{x\epsilon} s^n e^{-s}\,ds\right)\frac{(-1)^n}{x^n} \sim \sum_{n=0}^{\infty}\frac{(-1)^n n!}{x^{n+1}}$$

as exponentially small terms will always be dominated by a power of $(1/x)$ as $x\to\infty$.

Also $I_2 < \int_{\varepsilon}^{1} e^{-xt}\,dt = \underbrace{e^{-x\epsilon}}_{\text{already dropped terms this small}} - e^{-x}$ even smaller

$\ll I_1(x)$

$$\therefore I(x) \sim \sum_{n=0}^{\infty}\frac{(-1)^n n!}{x^{n+1}} \quad \text{as } x\to\infty$$

4.4 Watson's Lemma

Let $$I(x) = \int_0^b f(t) e^{-xt}\, dt \quad , \quad b>0,$$

with (i) $f(t)$ continuous on $t \in [0,b]$

(ii) If $b=\infty$, in addition $\exists c \in \mathbb{R}$ with $f(t) = o(e^{ct})$ as $t \to \infty$

(iii) $$f(t) \sim t^{\alpha} \sum_{n=0}^{\infty} a_n t^{\beta n} \quad \text{as } t \to 0^+$$

with $\alpha > -1$, $\beta > 0$, $a_n \in \mathbb{R}$ for $n \in \mathbb{N}_0$.

Then $$I(x) \sim \sum_{n=0}^{\infty} \frac{a_n \Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}} \quad \text{as } x \to \infty$$

where $$\Gamma(m) = \int_0^{\infty} t^{m-1} e^{-t}\, dt.$$

Note $\Gamma(m) = (m-1)!$ for $m \in \mathbb{N}$.

<u>Proof</u> See Supplementary Notes online.

{ If f uniformly convergent in neighbourhood of origin, proceeds as in example above

4.5 General Laplace Integrals

- Dominant contribution to

$$I(x) = \int_a^b f(t) e^{x\varphi(t)}\, dt \quad \text{as } x \to \infty$$

is from the region where $\varphi(t)$ is the largest.

- There are 3 cases: the maximum of $\varphi(t)$ is at (i) $t = a$, (ii) $t = b$, (iii) $t = c \in (a, b)$.

To proceed

- Isolate dominant contribution from near maximum of φ and reduce range of integration to this region
  - Gives exponentially small errors
- Taylor expand φ, f and rescale
- Finally extend range of integration once other approximations made

<u>Case (i)</u> with $\varphi'(a) < 0$, $f(a) \neq 0$, $\varphi''(a) \neq 0$

need to assess size of ε relative to $1/x$ --- later!

$$I(x) = \underbrace{\int_a^{a+\varepsilon} f(t) e^{x\varphi(t)}\, dt}_{I_1(x)} + \underbrace{\int_{a+\varepsilon}^b f(t) e^{x\varphi(t)}\, dt}_{I_2(x)}$$

$|I_1| \gg |I_2|$

$$e^{x\varphi(a+\varepsilon)} \ll e^{x\varphi(a)} \qquad \varphi(a+\varepsilon) \simeq \varphi(a) + \varepsilon\varphi'(a)$$

$$e^{x\varepsilon\varphi'(a)} \ll 1$$

$$\boxed{x\varepsilon \gg 1}$$

$$I_1(x) = \int_a^{a+\varepsilon} [f(a) + (t-a)f'(a) + \cdots] \exp\left[x\left\{\varphi(a) + (t-a)\varphi'(a) + \frac{(t-a)^2}{2}\varphi''(a) + \cdots\right\}\right] dt$$

$$= e^{x\varphi(a)} \int_a^{a+\varepsilon} [f(a) + (t-a)f'(a) + \cdots] e^{x(t-a)\varphi'(a)} \left[1 + x\frac{(t-a)^2}{2}\varphi''(a) + \cdots\right] dt$$

Rescale $x(t-a) = s$ ← Remove x from leading exponent.

okay given $\boxed{x\varepsilon^2 \ll 1}$ $\therefore$ $\boxed{\frac{1}{x} \ll \varepsilon \ll \frac{1}{\sqrt{x}}}$

$$= \frac{e^{x\varphi(a)}}{x} \int_0^{\varepsilon x} [f(a) + O(s/x)] e^{s\varphi'(a)} [1 + O(s^2/x)] ds$$

$\rightarrow\infty$ OKAY as $\varepsilon x \gg 1$

Explain in detail

$$= \frac{f(a)e^{x\varphi(a)}}{x} \left(\int_0^{\varepsilon x} e^{s\varphi'(a)} (1 + O(1/x)) \right) ds$$

$$= \frac{f(a)e^{x\varphi(a)}}{x|\varphi'(a)|} \left(1 + O(1/x)\right)$$

← guarantees asymptoticness... correction much smaller than last term.

$$\therefore \; I(x) \sim I_1(x) \sim \frac{f(a)e^{x\varphi(a)}}{x|\varphi'(a)|} \quad \text{as } x \to \infty.$$

<u>Case (ii)</u> with $\varphi'(b) > 0$, $f(b) \neq 0$, $\varphi''(b) \neq 0$. <u>Exercise</u> Show that

$$I(x) \sim \frac{f(b)e^{x\varphi(b)}}{x\,\varphi'(b)} \quad \text{as } x \to \infty.$$

← Essentially identical to case (i)

Case (iii) $\varphi'(c)=0, \quad \varphi''(c)<0, \quad f(c)\neq 0, \quad \varphi'''(c)\neq 0$

$t=c$ global maximum of $\varphi(t)$ for $t\in[a,b]$.

$$I(x) = \Bigg[\underbrace{\int_a^{c-\varepsilon} dt}_{I_1} + \underbrace{\int_{c-\varepsilon}^{c+\varepsilon} dt}_{I_2} + \underbrace{\int_{c+\varepsilon}^{b}}_{I_3}\Bigg] f(t)e^{x\varphi(t)}$$

I_2 dominant

$$e^{x\varphi(c+\varepsilon)} \ll e^{x\varphi(c)} \quad \text{for } |I_2| \gg |I_3|$$

$\varphi(c+\varepsilon) \approx \varphi(c) + \frac{\varepsilon^2}{2}\varphi''(c)$ as $\varphi'(c)=0$

$\therefore \quad e^{\frac{x\varepsilon^2\varphi''(c)}{2}} \ll 1 \quad \therefore \quad \boxed{x\varepsilon^2 \gg 1}$

Same argument for $|I_2| \gg |I_1|$

$$I_2(x) = \int_{c-\varepsilon}^{c+\varepsilon} dt\, f(t)e^{x\varphi(t)}$$

$$= \int_{c-\varepsilon}^{c+\varepsilon} \left[f(c) + O(t-c)\right] e^{x\varphi(c)}\, e^{x(t-c)^2/2\,\varphi''(c)} \cdot \Big[1 + \underbrace{O\big(x(t-c)^3/3!\big)}_{x\varepsilon^3 \ll 1}\Big]\, dt$$

$$\therefore \quad \underbrace{\boxed{\frac{1}{x^{1/2}} \ll \varepsilon \ll \frac{1}{x^{1/3}}}}_{\text{Need } x \text{ rather large}}$$

eg suppose $x=8$

$\frac{1}{2\sqrt{2}} \ll \varepsilon \ll 1/2$

but $\frac{1}{\sqrt{2}} \not\ll 1$

Rescale $s = \sqrt{x}(t-c)$

$$I_2(x) = \frac{f(c)e^{x\varphi(c)}}{\sqrt{x}} \int_{-\sqrt{x}\,\varepsilon}^{\sqrt{x}\,\varepsilon} ds \;\; e^{s^2/2\,\varphi''(c)} \underbrace{\left(1 + O\left(\frac{s}{\sqrt{x}}\right)\right)}_{\text{from expansion of } f} \underbrace{\left(1+O\left(\frac{s^3}{\sqrt{x}}\right)\right)}_{\text{from exponential expansion}}$$

$$= \frac{f(c)e^{x\varphi(c)}}{\sqrt{x}} \underbrace{\int_{-\infty}^{\infty} ds\, e^{s^2/2\,\varphi''(c)}}_{\sqrt{\frac{2}{-\varphi''(c)}}\int_{-\infty}^{\infty} du\, e^{-u^2}} \left(1+O\left(\frac{1}{\sqrt{x}}\right)\right)$$

okay as $\sqrt{x}\,\varepsilon \gg 1$

Substitute $-s^2/2\,\varphi''(c) = u^2$

$$= \sqrt{\frac{2}{-\varphi''(c)x}}\, f(c)e^{x\varphi(c)} \left(1+O\left(\frac{1}{\sqrt{x}}\right)\right)$$

$$\therefore \; I(x) \sim I_2(x) \sim \sqrt{\frac{2}{-\varphi''(c)x}}\, f(c)e^{x\varphi(c)} \quad \text{as } x \to \infty$$

4.6 Method of Stationary Phase

- Used when $\phi = i\psi$, ψ real, so that

$$I(x) = \int_a^b f(t)\, e^{ix\psi(t)}\, dt.$$

Riemann-Lebesgue Lemma

If $\int_a^b |f(t)|\, dt < \infty$ and $\psi(t)$ is continuously differentiable for $t \in [a,b]$ and not constant on any sub-interval of $[a,b]$

then $$\int_a^b f(t)\, e^{ix\psi(t)}\, dt \to 0 \quad \text{as } x \to \infty.$$

Useful

- Useful for integration by parts, eg.

$$I(x) = \int_0^1 \frac{e^{ixt}}{1+t}\, dt = \underline{-\frac{ie^{ix}}{2ix} + \frac{i}{x}} - \frac{i}{x} \underbrace{\int_0^1 \frac{e^{ixt}}{(1+t)^2}\, dt}_{\to 0 \text{ as } x\to\infty \text{ by RLL.}}$$

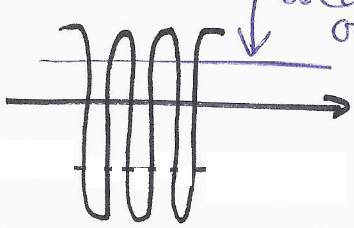
First term of an asymptotic expansion

- Why does RLL hold?

(i) For $\psi(t) = t$.

$$\int_a^b f(t)\, e^{ixt}\, dt$$

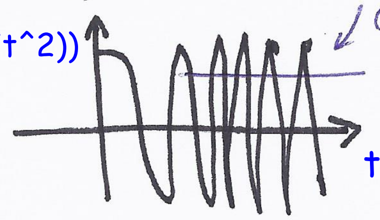
oscillates more and more rapidly



(ii) More generally.

Near $t = t_0$, $\psi(t) \sim \psi(t_0) + (t-t_0)\psi'(t_0) + \cdots$

Period of oscillation $\sim \dfrac{2\pi}{x|\psi'(t_0)|}$



cancellation as $x \to \infty$

$\to 0$ as $x \to \infty$ providing $|\psi'(t_0)| \neq 0$

$\therefore$ Again get cancellation, unless $|\psi'(t_0)| = 0$

Unless ψ is constant on a region of non-zero measure, a stationary point is not enough to save the integral as $x \to \infty$, and one gets zero.

Nonetheless the dominant terms for x large but not infinite are from where $|\psi'(t_0)| = 0$

Example

$\psi''(t) \sim \text{ord}(1)$ in neighbourhood of c.

$f(c) \neq 0$; $\psi'(c) = 0$, $c \in (a,b)$; $\psi'(t) \neq 0$ $t \in [a,b] \setminus \{c\}$.

$$I(x) = \left[\underbrace{\int_a^{c-\varepsilon}}_{I_1(x)} + \underbrace{\int_{c-\varepsilon}^{c+\varepsilon}}_{I_2(x)} + \underbrace{\int_{c+\varepsilon}^{b}}_{I_3(x)} \right] f(t) e^{ix\psi(t)} \, dt$$

$\varepsilon \ll 1$

$$I_2(x) = \int_{c-\varepsilon}^{c+\varepsilon} dt \left[f(c) + O(t-c)\right] \exp\left[ix\left\{\psi(c) + \tfrac{1}{2}(t-c)^2\psi''(c) + O\left((t-c)^3\right)\right\}\right].$$

1) Isolate dominant contribution (no longer need be a maximum) and reduce range of integration to this region.

need to check errors in the approx ... harder here ... will do this at the end

$$= e^{ix\psi(c)} \int_{c-\varepsilon}^{c+\varepsilon} dt \left[f(c)+O(t-c)\right] e^{\frac{ix}{2}(t-c)^2\psi''(c)} \left(1+O\left((t-c)^3 x\right)\right)$$

providing $\varepsilon^3 x \ll 1$

$\therefore \varepsilon \ll 1/x^{1/3}$

2) Taylor expand and rescale

$x(t-c)^2 = s^2$

$$= \frac{e^{ix\psi(c)}}{\sqrt{x}} \int_{-\varepsilon\sqrt{x}}^{\varepsilon\sqrt{x}} ds \left(f(c) + \underbrace{O\left(\tfrac{s}{\sqrt{x}}\right)}_{\text{subleading}}\right) e^{is^2\psi''(c)/2} \left(1+\underbrace{O\left(s^3/\sqrt{x}\right)}_{\text{subleading}}\right)$$

Drop

need to check scale of errors in the approximations ... will do this at the end

3) Extend range of integration

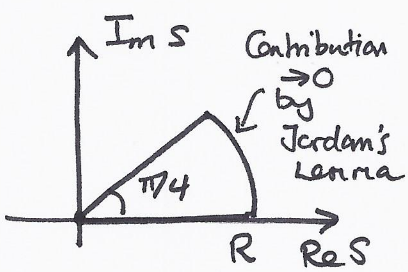
$$= \frac{e^{ix\psi(c)}f(c)}{\sqrt{x}} \int_{-\infty}^{\infty} ds\, e^{is^2\psi''(c)/2} + \dots$$

Requires $\varepsilon\sqrt{x} \gg 1$

$$\boxed{\frac{1}{x^{1/2}} \ll \varepsilon \ll \frac{1}{x^{1/3}}}$$

$$\int_{-\infty}^{\infty} ds\, e^{is^2\psi''(c)/2} = 2\int_0^{\infty} ds\, e^{is^2\psi''(c)/2}$$

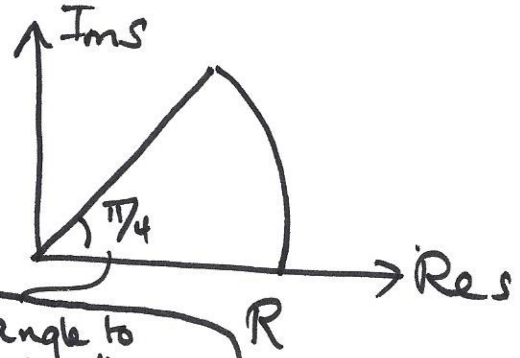
$$= \left(\frac{2\pi}{|\psi''(c)|}\right)^{1/2} e^{i\frac{\pi}{4}\,\mathrm{sgn}(\psi''(c))}$$



$\psi''(c) > 0$

Angle $-\pi/4$ for $\psi''(c) < 0$

With $\psi''(c) > 0$



take angle to be $-\pi/4$ if $\psi''(c) < 0$

$$s^2 = e^{i\pi/2} p \qquad s = e^{i\pi/4} p$$

Cauchy

$$0 = \oint_{\triangleleft R} ds\, e^{is^2\psi''(c)/2}$$

$$= \left[\int_{\rightarrow} + \int_{\swarrow}\right] ds\, e^{is^2\psi''(c)/2}$$

using $\int \to 0$ as $R \to \infty$ by Jordan's Lemma.

$$\therefore \int_0^\infty ds\, e^{is^2\psi''(c)/2} = \int_0^\infty dp\, e^{-p^2\psi''(c)/2} \cdot e^{i\pi/4}$$

$$= e^{i\pi/4}\sqrt{\frac{2\pi}{\psi''(c)}} \qquad \psi''(c) > 0$$

More generally

$$\int_0^\infty ds\, e^{is^2\psi''(c)/2} = e^{i\pi/4\,\mathrm{sgn}(\psi''(c))}\sqrt{\frac{2\pi}{|\psi''(c)|}}$$

$$\therefore\ I_2(x) = \frac{2\pi}{|\psi''(c)|^{1/2}} \exp\left[i\pi/4\ \mathrm{sgn}(\psi''(c))\right] \frac{e^{ix\psi''(c)}}{\sqrt{x}} f(c) + \ldots$$