$$\therefore\ I_2(x) = \frac{2\pi}{|\psi''(c)|^{1/2}} \exp\left[i\pi/4 \,\mathrm{sgn}(\psi''(c))\right] \frac{e^{ix\psi''(c)}}{\sqrt{x}} f(c) + \dots$$

Size of Correction terms

1) Corrections from change of limits

$$\int_{\varepsilon\sqrt{x}}^{\infty} e^{is^2\psi''(c)/2}\, ds = \int_{\varepsilon\sqrt{x}}^{\infty} \frac{ds}{is\psi''(c)} \, is\psi''(c) e^{is^2\psi''(c)/2}$$

$$= \left[\frac{1}{is\psi''(c)} e^{is^2\psi''(c)/2}\right]_{\varepsilon\sqrt{x}}^{\infty} - \underbrace{\int_{\varepsilon\sqrt{x}}^{\infty} \frac{-1}{is^2\psi''(c)} e^{is^2\psi''(c)/2}\, ds}_{\text{Smaller correction}}$$

$$= O\left(\frac{1}{\varepsilon\sqrt{x}}\right) \quad (\text{note } \varepsilon\sqrt{x} \gg 1).$$

Similar contribution from $\displaystyle\int_{-\infty}^{-\sqrt{x}\varepsilon} e^{is^2\psi''(c)/2}\, ds$

2) Corrections from Taylor Expansions

Expansion in $(t-c)^3 x = s^3/\sqrt{x}$

$$\underbrace{\frac{1}{\sqrt{x}} \int_{-\varepsilon\sqrt{x}}^{\varepsilon\sqrt{x}} \frac{s^n}{x^{n/2}} e^{is^2\psi''(c)/2}\, ds}, \qquad \underbrace{\frac{1}{\sqrt{x}} \int_{-\varepsilon\sqrt{x}}^{\varepsilon\sqrt{x}} \frac{(s^3)^n}{x^{n/2}} e^{is^2\psi''(c)/2}}$$

$\boxed{n \geq 1}$

$$\frac{1}{\sqrt{x}} \frac{1}{x^{n/2}} (\sqrt{x}\varepsilon)^{n-1} \sim \frac{\varepsilon^{n-1}}{x} \qquad \frac{1}{\sqrt{x}} \frac{1}{x^{n/2}} (\sqrt{x}\varepsilon)^{3n-1} \sim \frac{(\varepsilon^3 x)^n}{x\varepsilon} \sim \frac{\varepsilon(\varepsilon^3 x)^n}{x\varepsilon^2}$$

Using $\displaystyle\int_{-\varepsilon\sqrt{x}}^{\varepsilon\sqrt{x}} s^n e^{is^2\psi''(c)/2}\, ds = O\left((\sqrt{x}\varepsilon)^{n-1}\right)$ by parts.

3) <u>Correction from $I_1(x)$</u>

$$I_1(x) = \int_a^{c-\varepsilon} f(t) e^{ix\psi(t)} dt \qquad \frac{1}{x^{1/2}} \ll \varepsilon \ll 1/x^{1/3}$$

$$= \int_a^{c-\varepsilon} \frac{f(t)}{ix\psi'(t)} \frac{\partial}{\partial t}\left(e^{ix\psi(t)}\right) dt$$

$$= \left[\frac{f(t)}{ix\psi'(t)} e^{ix\psi(t)} \right]_a^{c-\varepsilon} \underbrace{- \frac{1}{ix} \underbrace{\int_a^{c-\varepsilon} e^{ix\psi(t)} \frac{\partial}{\partial t}\left(\frac{f(t)}{\psi'(t)} \right) dt}_{\to 0 \text{ as } x\to\infty \text{ by RLL}}}_{\text{if it exists}}$$

$$\sim O\left(\frac{1}{x\psi'(c-\varepsilon)} \right) \qquad o\left(\frac{1}{x}\right)$$

Small "oh"

$$\sim O\left(\frac{1}{\varepsilon x} \right)$$

$\psi'' \sim O(1)$ in neighbourhood of c.

Similarly for I_3.

<u>Note</u> Corrections algebraically small, not exponentially small as in other methods

Next order terms very difficult to find ⟩ in general need to consider whole integration domain not just behaviour near $t=c$

$$\boxed{\therefore\ I(x) \sim \frac{2\pi}{|\psi''(c)|^{1/2}} e^{i\pi/4 \,\mathrm{sgn}(\psi''(c))} \frac{e^{ix\psi(c)}}{\sqrt{x}} f(c)}$$

with corrections at $O\left(\frac{1}{\varepsilon\sqrt{x}}\right)$

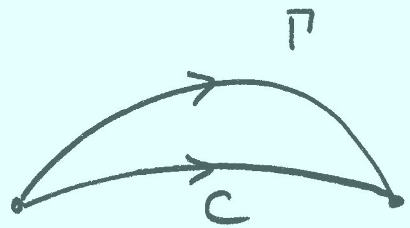
4.7 Method of Steepest Descents

- Generalises Laplace's method to consider

$$I(x) = \int_C f(t)\, e^{x\varphi(t)}\, dt \qquad \text{as } x \to \infty,\ x \text{ real},$$

where $f(t), \varphi(t)$ are holomorphic (and thus analytic), with C a contour in the complex t plane.

- Key idea: $I(x)$ unchanged upon deforming C to a new contour Γ, with the same start and end points.



$$I(x) = \int_\Gamma f(t)\, e^{x\varphi(t)}\, dt$$

- If we find a contour Γ on which $\mathrm{Im}(\varphi(t))$ is piecewise constant, i.e. Γ_j, v_j such that $\Gamma = \bigcup_j \Gamma_j$ with $\mathrm{Im}\,\varphi(t) = v_j = \text{const}$ on Γ_j then

$$I(x) = \sum_j e^{ixv_j} \int_{\Gamma_j} f(t) e^{x \operatorname{Re} \varphi(t)} dt$$

and each integral can be analysed as $x \to \infty$ using Laplace's method.

· Let $\varphi(t) = u(\xi, \eta) + iv(\xi, \eta)$ with $t = \xi + i\eta$.

· As φ is holomorphic, we have the Cauchy Riemann Equations (CRE):

$$\frac{\partial u}{\partial \xi} = \frac{\partial v}{\partial \eta}, \quad \frac{\partial u}{\partial \eta} = -\frac{\partial v}{\partial \xi}.$$

Hence $\nabla u \cdot \nabla v = u_\xi v_\xi + u_\eta v_\eta = 0$ $\therefore$ $\nabla u \perp \nabla v$

Also $\nabla v \perp$ contours with v const $\therefore$ Contours with v const $/\!/ \ \nabla u$.

∇u points in direction u increases at fastest rate

$-\nabla u$ points in direction u decreases at fastest rate

$\therefore$ Contour with v constant is a path of steepest ascent/descent of u.

Landscape of $u(\xi,\eta)$

- CRE. $\quad u_{\xi\xi}+u_{\eta\eta} = (v_\eta)_\xi + (-v_\xi)_\eta = 0$

- Hence u cannot have a maximum or a minimum (unless we are also considering a point where u is singular or a branch point, where u is not holomorphic).

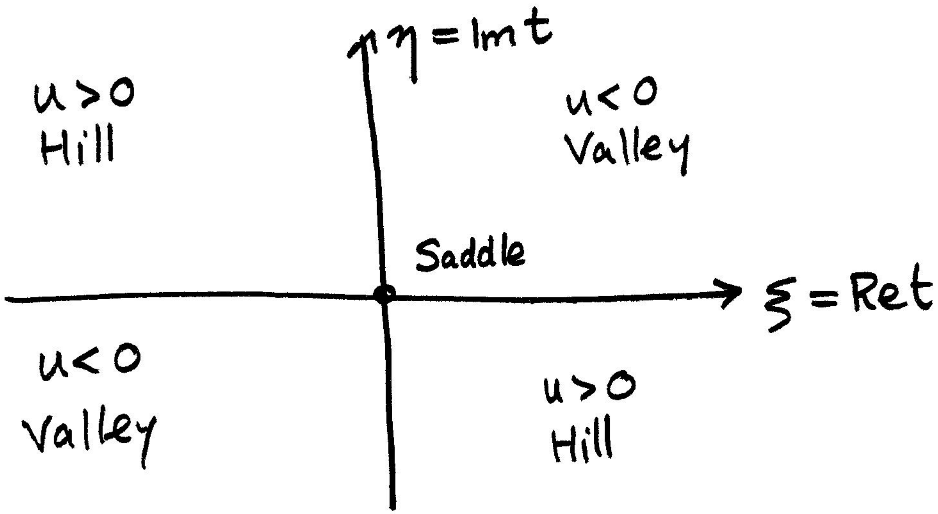
- At a stationary point, where $u_\xi = u_\eta = 0$, we have a SADDLE.

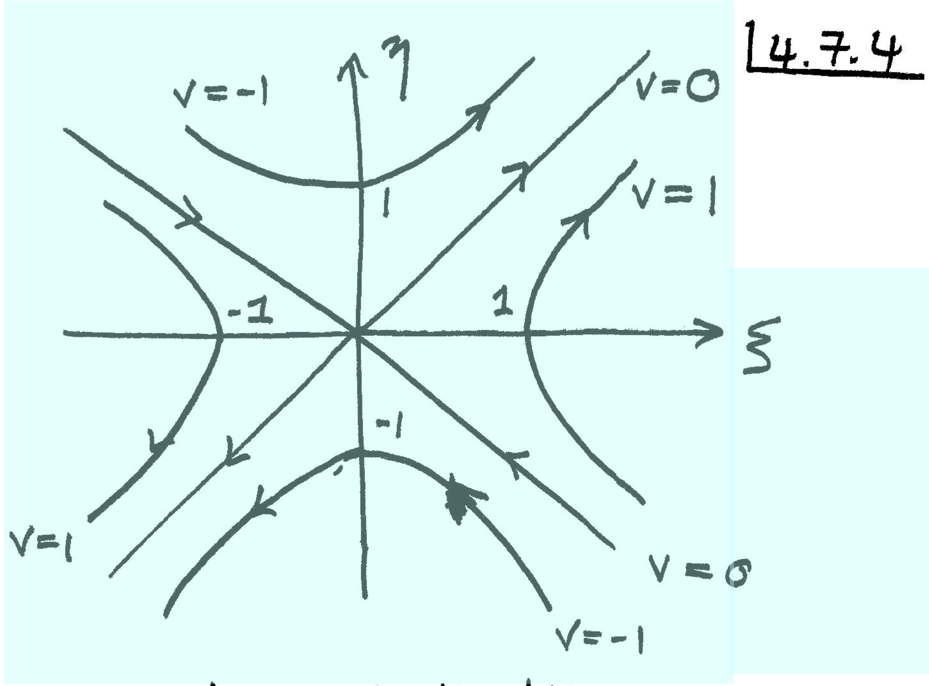
- Landscape of u has hills $(u>0)$, valleys $(u<0)$ at infinity with saddle points in the interior of the complex plane.

Example

$$\varphi(t) = it^2 = i(\xi+i\eta)^2 = -2\xi\eta + i(\xi^2-\eta^2) \quad \therefore\ u = -2\xi\eta\,,\ v = \xi^2-\eta^2$$

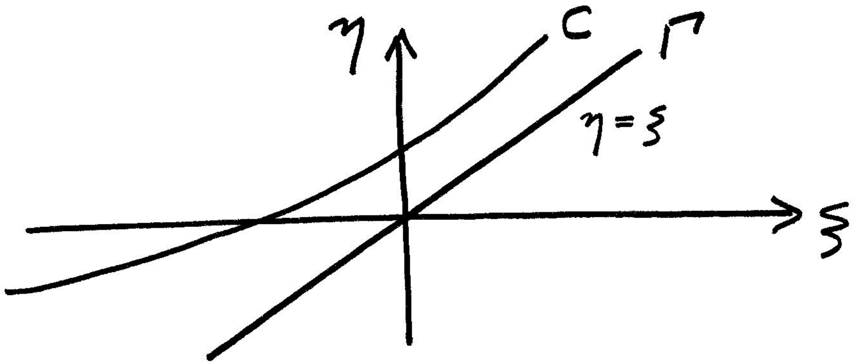
$$\nabla u = -2(\eta,\xi) \quad \therefore \text{ Saddle point at } \xi=\eta=0$$





Arrows in direction of decreasing u with STEEPEST DESCENT

- Contour C infinite, with endpoints in different valleys.
  - If endpoints not in valleys, integral $I(x)$ not well defined.



Deform C into Γ.

Integrals at infinity subleading

Hence method known as "Method of steepest descents" or saddle point method

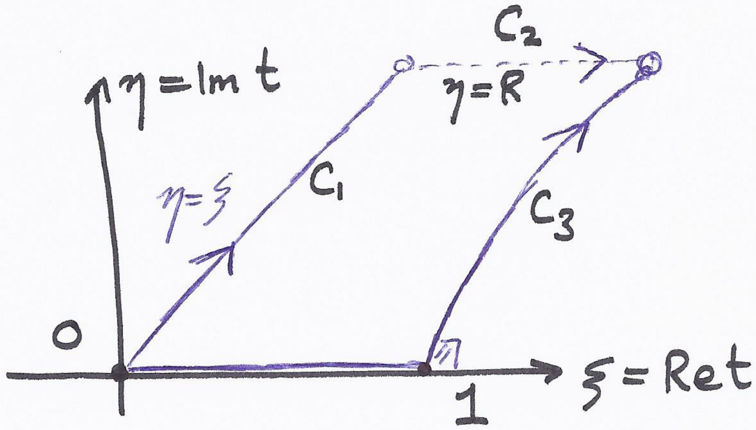
To use the method ...

* Deform contour to union of steepest descent (v const) contours through the endpoints and any relevant saddle points

* Evaluate local contributions from saddle and end points using Laplace's method.

Example

$$I(x) = \int_0^1 e^{x\varphi(t)}\,dt \quad \text{as } x \to \infty, \text{ with } \varphi(t) = it^2.$$



Steepest descent contour through $t=0$ is $\eta = \xi$

Steepest descent contour through $t = 1$ is $\xi^2 - \eta^2 = 1$

$$C_1(R) = \{ \xi(1+i),\ \ \xi \in [0, R] \}$$

$$C_2(R) = \{ \xi + iR,\ \ \xi \in [R, \sqrt{R^2+1}\,] \}$$

$$C_3(R) = \{ \sqrt{1+\eta^2} + i\eta,\ \ \eta \in [0, R] \}$$

$$\therefore\ I(x) = \left[\int_{C_1(R)} + \int_{C_2(R)} - \int_{C_3(R)} \right] e^{ixt^2}\,dt$$

On $C_2(R)$ $\quad |\exp(ixt^2)| = |\exp(ix(\xi^2 - R^2 + 2i\xi R))|$

$$= |\exp(-2x\xi R)| = O(e^{-2xR^2}) \to 0$$

as $R \to \infty$.

$$\therefore\ \int_{C_2(R)} e^{ixt^2}\,dt \to 0 \quad \text{as } R \to \infty.$$

$$\int_{C_1(\infty)} e^{ixt^2}dt = \int_0^\infty \exp\left(ix\xi^2(1+i)^2\right) d\xi\,(1+i)$$

$i(1+i)^2 = i(1+2i+i^2)$
$= 2i^2 = -2.$

$$= (1+i)\int_0^\infty e^{-2x\xi^2}d\xi \qquad u = \sqrt{2x}\,\xi$$

$$= \frac{1+i}{\sqrt{2}\sqrt{x}}\int_0^\infty e^{-u^2}du = \frac{e^{i\pi/4}}{2}\sqrt{\frac{\pi}{x}}.$$

$$\int_{C_3(\infty)} e^{ixt^2}dt = \int_0^\infty e^{ix\underbrace{\left[(1+\eta^2)^{1/2}+i\eta\right]^2}_{1+2i\eta(1+\eta^2)^{1/2}}}\frac{dt}{d\eta}\,d\eta$$

$$= e^{ix}\int_0^\infty e^{x\varphi(\eta)}f(\eta)\,d\eta$$

with $\varphi(\eta) = -2\eta(1+\eta^2)^{1/2}$,

$$f(\eta) = \frac{dt}{d\eta} = \frac{\eta}{(1+\eta^2)^{1/2}} + i$$

and thus Laplace's method can be used.

However, we can get to a quicker answer, at all orders, by noting

on $C_3(\infty)$, $\quad t = \xi + i\eta \quad$ where $\xi^2 - \eta^2 = 1$

$$\therefore\ t^2 = \xi^2 - \eta^2 + 2i\xi\eta = 1 + 2i\eta(1+\eta^2)^{1/2}$$

$\therefore$ Let $t^2 = 1 + is \qquad s \in [0,\infty)$

$\therefore$ $\underline{\underline{t = (1+is)^{1/2}}}$ (principal branch ie +ve squareroot).

$\boxed{4.7.8}$

$\dfrac{dt}{ds} = \frac{1}{2} i \dfrac{1}{(1+is)^{1/2}}$

Then

$$\int_{C_3(\infty)} e^{ixt^2}\,dt = \int_0^\infty e^{ix} \cdot e^{-xs} \frac{dt}{ds}\,ds$$

$$= \frac{ie^{ix}}{2} \int_0^\infty e^{-xs} \frac{1}{(1+is)^{1/2}}\,ds$$

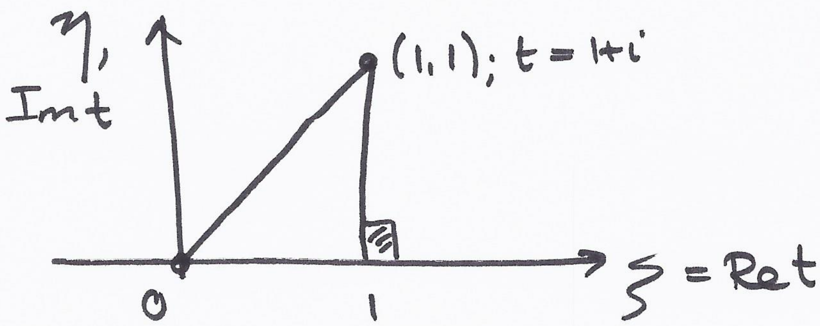
$$\underset{\text{Lemma}}{\overset{\text{Watson's}}{\sim}} \frac{ie^{ix}}{2} \sum_{n=0}^{\infty} \frac{a_n \Gamma(n+1)}{x^{n+1}} \quad \text{as } x \to \infty$$

$$\text{with } a_n = \frac{(-i)^n \Gamma(n+\frac{1}{2})}{\Gamma(n+1)\sqrt{\pi}}$$

$$\therefore I(x) \sim \frac{e^{i\pi/4}}{2}\sqrt{\pi/x} - \frac{ie^{ix}}{2\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-i)^n \Gamma(n+\frac{1}{2})}{x^{n+1}} \quad \text{as } x \to \infty$$

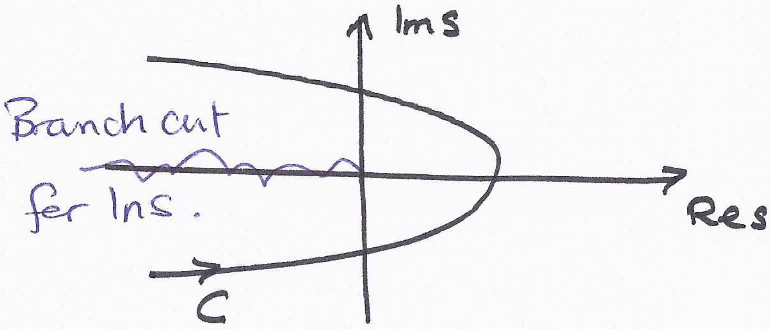
Note

Local contributions dominate ... just need to get tangents to steepest descent paths ... eg. could use



in the above example.

Example



$$I(x) = \int_C e^s s^{-x}\, ds$$

as $x \to \infty$

Note $\quad e^s s^{-x} = \exp\left[s - x\log s\right]$, branch cut for $\log s$, is given by $\{\operatorname{Re} s < 0, \operatorname{Im} s = 0\}$

Saddle point at $x/s = 1$.
Fix saddle point location by setting $t = s/x$.

Let $s = tx$

$$I(x) = x\int_{C_*} dt\, \underbrace{e^{tx - x\log(tx)}}_{e^{tx - x\log t - x\log x}} = x^{1-x}\int_{C_*} dt\, e^{x\varphi(t)}$$

with $\varphi(t) = t - \log t$.

$\therefore \varphi = \xi + i\eta - \log r - i\theta$

polars.

$0 = \varphi'(t) = 1 - 1/t \quad \therefore$ Saddle at $t = 1$

Deform C_* through this point

$$u = \operatorname{Re}\varphi = r\cos\theta - \log r \qquad v = \operatorname{Im}\varphi = r\sin\theta - \theta$$

At $t=1$ $\quad \theta = 0, \; v = 0$

$\therefore$ Path of steepest descent through $t=1$ given by

$$r = \frac{\theta}{\sin\theta} \qquad \theta \in (-\pi, \pi)$$

On this path, Γ

$$u = \operatorname{Re}\varphi = r(\theta)\cos\theta - \log r(\theta)$$
$$= \theta\cot\theta - \log\theta + \log\sin\theta$$