4.7.10

$$\therefore\ I(x) = x^{1-x}\int_{-\pi}^{\pi} e^{x u(\theta)}\,\frac{dt}{d\theta}\,d\theta$$

$t = r(\theta)e^{i\theta}$

$dt/d\theta = (r'(\theta) + i r(\theta))\, e^{i\theta}$

$$= x^{1-x}\int_{-\pi}^{\pi} d\theta\; e^{x\{\underbrace{\theta\cot\theta - \log\left(\frac{\theta}{\sin\theta}\right)}_{\Psi(\theta)}\}}\underbrace{\left[r'(\theta)+ir(\theta)\right]e^{i\theta}}_{F(\theta)}$$

Laplace's method with interior maximum at $\theta = 0$

$$I(x) \sim x^{1-x}\,\frac{\sqrt{2\pi}\,F(0)e^{x\Psi(0)}}{\sqrt{-\Psi''(0)x}} \qquad \text{as } x\to\infty$$

By Taylor expanding, $r(\theta) = \dfrac{\theta}{\sin\theta} = \dfrac{\theta}{\theta - \theta^3/3! + \cdots}$ $= 1+\theta^2/6+O(\theta^3)$

and hence $F(0) = i$

$$\Psi(\theta) = \frac{\theta(1-\theta^2/2!+\cdots)}{\theta-\theta^3/3!+\cdots} - \log\left(1+\theta^2/6+O(\theta^3)\right)$$

$$= 1 - \theta^2/2 + O(\theta^3)$$

$\Psi(0) = 1 \qquad \Psi''(0) = -1$

$$\therefore\ I(x) \sim i\,x^{1/2-x}e^{x}\sqrt{2\pi} \quad \text{as } x\to\infty$$

NB this example can be used to deduce $\Gamma(x) \sim \sqrt{2\pi}\,x^{x-1/2}e^{-x}$,
ie. Stirling's approx... see online notes.

4.8 Splitting Integration Range

* Previously, have split integration range to isolate dominant contribution
* More generally, can split integration range and use different approximations in each range

$x \sim O(1)$ Integrand $O(1)$
Integration range $O(1)$
Integral $O(1)$

Do not lecture this example.

Example $$I(\varepsilon) = \int_0^1 \frac{dx}{(x+\varepsilon)^{1/2}} \quad \text{as } \varepsilon \to 0^+.$$

$x \sim O(\varepsilon)$ Integrand $O(1/\varepsilon^{1/2})$

$x = \text{ord}(1)$
$$\frac{1}{(x+\varepsilon)^{1/2}} = \frac{1}{x^{1/2}} \frac{1}{(1+\varepsilon/x)^{1/2}}$$
$$= \frac{1}{x^{1/2}}\left(1 - \frac{\varepsilon}{2x} + O\left(\varepsilon^2/x^2\right)\right)$$
as $\varepsilon \to 0$

Expansion not valid for $x \sim O(\varepsilon)$

$\therefore$ Split.
$$I(x) = \underbrace{\int_0^\delta \frac{dx}{(x+\varepsilon)^{1/2}}}_{I_1} + \underbrace{\int_\delta^1 \frac{dx}{(x+\varepsilon)^{1/2}}}_{I_2} \qquad \varepsilon \ll \delta \ll 1$$

$O(1)$

$$I_2 = \int_\delta^1 dx\left(\frac{1}{x^{1/2}} - \frac{\varepsilon}{2x^{3/2}} + O\left(\varepsilon^2/x^{5/2}\right)\right)$$
okay as $\frac{\varepsilon}{x} < \varepsilon/\delta \ll 1$

$$= 2(1-\delta^{1/2}) + \varepsilon\left(1 - \frac{1}{\sqrt{\delta}}\right) + O\left(\varepsilon^2/\delta^{3/2}\right)$$

$$I_1 = \int_0^{\delta} \frac{dx}{(x+\varepsilon)^{1/2}} \qquad \text{Let } x = \varepsilon u$$

$$= \int_0^{\delta/\varepsilon} \frac{\varepsilon\, du}{\varepsilon^{1/2}(1+u)^{1/2}} = 2\varepsilon^{1/2}\left(1+\delta/\varepsilon\right)^{1/2} - 2\varepsilon^{1/2}$$

$\varepsilon/\delta \ll 1$

$$= 2\delta^{1/2}\left(1+\varepsilon/\delta\right)^{1/2} - 2\varepsilon^{1/2}$$

$$= 2\delta^{1/2} + \varepsilon/\delta^{1/2} + O\left(\varepsilon^2/\delta^{3/2}\right) - 2\varepsilon^{1/2}$$

$$\therefore I = I_1 + I_2 = 2 - 2\delta^{1/2} + \varepsilon - \varepsilon/\delta^{1/2} + O\left(\varepsilon^2/\delta^{3/2}\right)$$
$$+ 2\delta^{1/2} + \varepsilon/\delta^{1/2} - 2\varepsilon^{1/2} + O\left(\varepsilon^2/\delta^{3/2}\right)$$

$$= 2 - 2\varepsilon^{1/2} + \varepsilon + \cdots \qquad \text{noting } \varepsilon \ll \delta$$

$$\therefore \frac{\varepsilon^2}{\delta^{3/2}} = \frac{\varepsilon^2}{\delta^2}\delta^{1/2} \ll 1$$

<u>NB</u> Exact value

$$I(\varepsilon) = 2\left((1+\varepsilon)^{1/2} - \varepsilon^{1/2}\right) = 2 - 2\varepsilon^{1/2} + \varepsilon + \cdots\cdot$$

<u>Example</u>

$$I(\varepsilon) = \int_0^{\pi/4} \frac{d\theta}{\varepsilon^2 + \sin^2\theta} \qquad \text{as } \varepsilon \to 0^+$$

$\theta \sim O(1)$ Integrand $\sim O(1)$ Integral $\sim O(1)$

$\theta \sim O(\varepsilon)$ Integrand $\sim O(\frac{1}{\varepsilon^2})$ Integration range $\sim O(\varepsilon)$

Integral $\sim O(1/\varepsilon)$

<u>Split</u>

$$I = \underbrace{\int_0^{\delta} \frac{d\theta}{\varepsilon^2 + \sin^2\theta}}_{I_1} + \underbrace{\int_{\delta}^{\pi/4} \frac{d\theta}{\varepsilon^2+\sin^2\theta}}_{I_2} \qquad \varepsilon \ll \delta \ll 1$$

need this to be small ... thus need

$$I_2 = \int_{\delta}^{\pi/4} \left(\frac{1}{\sin^2\theta} + O\left(\frac{\varepsilon^2}{\sin^4\theta}\right)\right) d\theta$$

$\varepsilon^2/\delta^4 \ll 1$

i.e. $\varepsilon^{1/2} \ll \delta \ll 1$

$$= -\left[\cot\theta\right]_{\delta}^{\pi/4} + O(\varepsilon^2/\delta^3)$$

$$= -1 + \frac{(1-\delta^2/2+\dots)}{\delta - \delta^3/6 + \dots} + O(\varepsilon^2/\delta^3) = -1 + \frac{1}{\delta} + O(\delta) + \underbrace{O(\varepsilon^2/\delta^3)}$$

$\ll 1$ for $\varepsilon^{2/3} \ll \delta \ll 1$

consistent with $\varepsilon^{1/2} \ll \delta \ll 1$

wlog $\varepsilon > 0$ $\quad \theta = \varepsilon u$ $\qquad \varepsilon u \le \varepsilon \cdot \frac{\delta}{\varepsilon} = \delta \ll 1$

$$I_1 = \int_0^{\delta/\varepsilon} \frac{\varepsilon \, du}{\varepsilon^2 + \sin^2(\varepsilon u)}$$

$$= \varepsilon \int_0^{\delta/\varepsilon} \frac{du}{\varepsilon^2 + \varepsilon^2 u^2 + O(\varepsilon^4 u^4)}$$

$$I_1 = {}^1\!/\!_\varepsilon \int_0^{\delta/\varepsilon} du \left[\frac{1}{1+u^2} + O\left(\frac{\varepsilon^2 u^4}{(1+u^2)^2} \right) \right]$$

$$= \frac{1}{\varepsilon} \tan^{-1}(\delta/\varepsilon) + O\left(\frac{1}{\varepsilon} \cdot \frac{\delta}{\varepsilon} \cdot \varepsilon^2 \right)$$

$$= \frac{1}{\varepsilon} \left[\pi/2 - \frac{1}{\delta/\varepsilon} + O\left(\frac{1}{(\delta/\varepsilon)^2} \right) \right] + O(\delta)$$

$$= \frac{\pi}{2\varepsilon} - \frac{1}{\delta} + \underbrace{O\left(\varepsilon/\delta^2 \right)}_{\ll 1 \text{ for } \varepsilon^{1/2} \ll \delta \ll 1} + O(\delta)$$

$$\therefore \quad I = I_1 + I_2 = -1 + 1/\delta + \frac{\pi}{2\varepsilon} - \frac{1}{\delta} + \underbrace{O\left(\frac{\varepsilon^2}{\delta^3}, \frac{\varepsilon}{\delta^2}, \delta \right)}_{\ll 1}$$

$$= \frac{\pi}{2\varepsilon} - 1 + \cdots \quad \text{as } \varepsilon \to 0$$

5. Matched Asymptotic Expansions

5.1 Singular Perturbations

Example $\qquad \varepsilon y'' + y' + y = 0 \qquad 0<x<1, \quad y(0)=a, \; y(1)=b.$

$\varepsilon = 0$ $\qquad y' + y = 0$. Hence $y = Ae^{-x}$, which cannot satisfy both boundary conditions in general.

This is a singular perturbation problem.

More generally suppose D_ε is a differential operator that depends on a small parameter ε, e.g. $D_\varepsilon = \varepsilon \, d^2/dx^2 + d/dx + 1$.

Then a differential equation $D_\varepsilon y = 0$ with boundary conditions is a singular perturbation problem if the order of $D_0 y$ is less than the order of $D_\varepsilon y$ as $\varepsilon \to 0$

[Since the solution of $D_0 y$ cannot satisfy BCs in general].

Suppose $D_\varepsilon = \varepsilon \dfrac{d^k}{dx^k}$ + lower order derivatives.

* Over most of the range, $\varepsilon \dfrac{d^k y}{dx^k}$ is small and y satisfies $D_0 y = 0$ to good approximation.

* In some regions, typically near boundaries, $\varepsilon \dfrac{d^k y}{dx^k}$ is not small and y adjusts to satisfy BCs.

The usual procedure for finding a solution to a singular ODE problem is:

(*) Determine the scaling in the boundary layers eg.

$$x = \varepsilon \hat{x} \quad \text{or} \quad x = \varepsilon^{1/2} \hat{x}$$

(*) Find the asymptotic expansions in the boundary layers ("<u>inner</u>" solution) and outside the boundary layers ("<u>outer</u>" solutions).

(*) Fix the constants of integration in these solutions by

- demanding the inner solutions satisfy the BCs
- "<u>matching</u>" - ensuring the expansion of the inner and outer solutions agree in an overlap region between them.

This is the method of <u>Matched Asymptotic Expansions</u>

<u>Previous Example</u> $\varepsilon y'' + y' + y = 0 \qquad 0 < x < 1, \quad y(0) = a, \; y(1) = b.$

Left Hand Boundary Scaling

Let $x = \varepsilon^{\alpha} x_L \quad y(x) = y_L(x_L)$ with $\alpha > 0$.

$\therefore \quad dy/dx = \frac{1}{\varepsilon^{\alpha}} \frac{dy_L}{dx_L}$ and $\quad \varepsilon^{1-2\alpha} \frac{d^2 y_L}{dx_L} + \varepsilon^{-\alpha} \frac{dy_L}{dx_L} + y_L = 0$

Dominant balance $\quad 1-2\alpha = -\alpha \quad \therefore \alpha = 1$. Hence boundary layer has width of $\mathrm{ord}(\varepsilon)$.

Right Hand Boundary Layer: Proceeds similary with $x = 1 + \varepsilon^{\beta} x_R$, $y(x) = y_R(x_R)$.
one finds $\beta = 1$.

Develop asymptotic solution

(1) Away from boundary layers (outer region), expand $y(x) \sim y_{out,0}(x) + \varepsilon y_{out,1}(x) + \ldots$
as $\varepsilon \to 0^+$ with $x, 1-x = \mathrm{ord}(1)$

(2) Left Hand Boundary. Let $x = \varepsilon x_L$ and expand

$y(x) = y_L(x_L) \sim y_{L,0}(x_L) + \varepsilon y_{L,1}(x_L) + \cdots$ as $\varepsilon \to 0^+$ with $x_L = \mathrm{ord}(1)$.

(3) Right hand boundary. Let $x = 1+\varepsilon x_R$ and expand

$$y(x) = y_R(x_R) \sim y_{R,0}(x_R) + \varepsilon y_{R,1}(x_R) + \cdots \quad \text{as } \varepsilon \to 0^+ \text{ with } -x_R \sim \text{ord}(1)$$

Left hand boundary layer

$$\frac{d^2 y_L}{dx_L^2} + \frac{dy_L}{dx_L} + \varepsilon y_L = 0 \quad , \quad x_L > 0 .$$

$O(\varepsilon^0)$ $$\frac{d^2 y_{L,0}}{dx_L^2} + \frac{dy_{L,0}}{dx_L} = 0 , \quad x_L > 0.$$

$O(\varepsilon^1)$ $$\frac{d^2 y_{L,1}}{dx_L^2} + \frac{dy_{L,1}}{dx_L} + y_{L,0} = 0, \quad x_L > 0.$$

$$\therefore \quad y_{L,0} = A_{L,0} + B_{L,0} e^{-x_L}$$

$$y_{L,1} = A_{L,1} + B_{L,1} e^{-x_L} + \left(B_{L,0} x_L e^{-x_L} - A_{L,0} x_L \right)$$

BC $\quad y_{L,0}(0) = a, \quad y_{L,1}(0) = 0 \quad \therefore \quad A_{L,0} + B_{L,0} = a, \quad A_{L,1} + B_{L,1} = 0.$

Right hand boundary layer

$$\frac{d^2 y_R}{dx_R^2} + \frac{dy_R}{dx_R} + \varepsilon y_R = 0 \qquad x_R < 0$$

As with left hand layer
$$y_{R,0}(x_R) = A_{R,0} + B_{R,0} e^{-x_R} \qquad (x_R < 0)$$
$$y_{R,1}(x_R) = A_{R,1} + B_{R,1} e^{-x_R} + \left(B_{R,0}\, x_R e^{-x_R} - A_{R,0}\, x_R\right)$$

with $A_{R,0} + B_{R,0} = b$, $A_{R,1} + B_{R,1} = 0$

Outer region

$$\varepsilon \frac{d^2 y_{out}}{dx^2} + \frac{dy_{out}}{dx} + y_{out} = 0 \qquad 0 < x < 1$$

$O(\varepsilon^0)$
$$\frac{dy_{out,0}}{dx} + y_{out,0} = 0$$

$O(\varepsilon^1)$
$$\frac{dy_{out,1}}{dx} + y_{out,1} = -\frac{d^2 y_{out,0}}{dx^2}$$

Solve

$$y_{out,0} = A_{out,0} e^{-x}$$

$$y_{out,1} = A_{out,1} e^{-x} - A_{out,0}\, x e^{-x}$$

Instead of applying BCs at $x = 0, 1$, we need to match with the left and right boundary layer solutions

Idea: There is an overlap, or intermediate, region where both expansions hold and therefore are equal.

Hence Introduce an intermediate scaling, $x = \varepsilon^{\gamma}\hat{x}$ with $0 < \gamma < 1$. Then with $\hat{x} > 0$, $\hat{x} = \mathrm{ord}(1)$

$$x = \varepsilon^{\gamma}\hat{x} \to 0, \quad x_L = \varepsilon^{\gamma-1}\hat{x} \to \infty \quad \text{as } \varepsilon \to 0^+.$$

Matching requires expansions to be equal as $\varepsilon \to 0^+$ with $\hat{x} > 0$, $\hat{x} = \mathrm{ord}(1)$

i.e. $$y_L(\varepsilon^{\gamma-1}\hat{x}) \sim y_{out}(\varepsilon^{\gamma}\hat{x}) \quad \text{as } \varepsilon \to 0^+ \text{ with } \hat{x} > 0, \hat{x} = \mathrm{ord}(1)$$

We have

$$y_L(\varepsilon^{\gamma-1}\hat{x}) = A_{L,0} + \underbrace{B_{L,0}\, e^{-\varepsilon^{\gamma-1}\hat{x}}}_{\text{exponentially small}} + O(\varepsilon)$$

$$y_{out}(\varepsilon^{\gamma}\hat{x}) = A_{out,0}\, e^{-\varepsilon^{\gamma}\hat{x}} + O(\varepsilon) = A_{out,0}\left(1 - \varepsilon^{\gamma}\hat{x} + O(\varepsilon^{2\gamma})\right) + O(\varepsilon)$$

Same expansions $\qquad A_{L,0} = A_{out,0} \qquad$ i.e. $\qquad y_{L,0}(\infty) = y_{out,0}(0)$

Matching outer and right hand boundary layer

Let $x = 1 + \varepsilon^{\gamma}\hat{x}$ with $0<\gamma<1$. As $\varepsilon \to 0^+$, with $\hat{x}<0$ and $\hat{x} = \mathrm{ord}(1)$

$$y_R(x_R = \varepsilon^{\gamma-1}\hat{x}) = A_{R,0} + \underbrace{B_{R,0}\, e^{-\varepsilon^{\gamma-1}\hat{x}}}_{\text{exponential blow up as } \varepsilon\to 0^+} + O(\varepsilon)$$

$$y_{out}(x = 1+\varepsilon^{\gamma}\hat{x}) = A_{out,0}\, e^{-(1+\varepsilon^{\gamma}\hat{x})} + O(\varepsilon)$$

$$= \frac{A_{out,0}}{e}\left(1 - \varepsilon^{\gamma}\hat{x} + O(\varepsilon^{2\gamma})\right) + O(\varepsilon)$$

Same expansions : $B_{R,0} = 0, \quad A_{out,0} = e A_{R,0}$

$$\therefore \left\{\begin{array}{l} A_{L,0} + B_{L,0} = a;\ \ A_{R,0} + B_{R,0} = b \\ A_{L,0} = A_{out,0};\ B_{R,0} = 0;\ A_{out,0} = e A_{R,0} \end{array}\right\} \therefore \left\{\begin{array}{l} A_{L,0} = eb \ ;\ A_{out,0} = eb \\ B_{L,0} = a - eb;\ A_{R,0} = b;\ B_{R,0} = 0 \end{array}\right\}$$

$$\therefore \quad y_{L,0}(x_L) = eb + (a-eb)e^{-x_L}\ ;\ y_{out,0}(x) = ebe^{-x}\ ;\ y_{R,0}(x_R) = b.$$

Agreement with exact solution

Exact solution is $y(x) = A_+ e^{\lambda_- x} - A_- e^{\lambda_+ x}$ for $0 \le x \le 1$

with $A_\pm = \dfrac{a e^{\lambda_\pm} - b}{e^{\lambda_+} - e^{\lambda_-}}$, $\lambda_\pm = \dfrac{-1 \pm \sqrt{1-4\varepsilon}}{2\varepsilon}$

Using expansions $\lambda_+ = -1 + O(\varepsilon)$; $\lambda_- = -\frac{1}{\varepsilon} + 1 + O(\varepsilon)$ as $\varepsilon \to 0^+$

can show

$y(\varepsilon x_L) = y_{L,0}(x_L) + O(\varepsilon)$ as $\varepsilon \to 0^+$ with $x_L > 0$, $x_L = \mathrm{ord}(1)$

$y(x) = y_{out,0}(x) + O(\varepsilon)$ as $\varepsilon \to 0^+$ with $0 < x < 1$ with $x, 1-x = \mathrm{ord}(1)$

$y(1+\varepsilon x_R) = y_{R,0}(x_R) + O(\varepsilon)$ as $\varepsilon \to 0^+$ with $x_R < 0$ $x_R = \mathrm{ord}(1)$.

Higher order Matching

Using the leading order solution, the first higher order solution is given by

$$y_{L,1}(x_L) = -eb\,x_L + (a-eb)\,x_L e^{-x_L} + A_{L,1} + B_{L,1} e^{-x_L}$$

$$y_{R,1}(x_R) = -b\,x_R + A_{R,1} + B_{R,1} e^{-x_R}$$

$$y_{out,1}(x) = -eb\,x e^{-x} + A_{out,1} e^{-x}$$

Recall BCs $\quad y_{L,1}(0)=0 \quad y_{R,1}(0)=0 \quad \therefore A_{L,1}+B_{L,1}=A_{R,1}+B_{R,1}=0$

Matching Left hand boundary layer and outer region

As $\varepsilon\to 0^+$ with $\hat{x}>0$ $\hat{x}=\mathrm{ord}(1)$ where $x=\varepsilon^{\gamma}\hat{x}$, $0<\gamma<1$

$$y_L(x_L=\varepsilon^{\gamma-1}\hat{x}) = y_{L,0}(\varepsilon^{\gamma-1}\hat{x}) + \varepsilon y_{L,1}(\varepsilon^{\gamma-1}\hat{x}) + O(\varepsilon^2)$$

$$= \left(eb + (a-eb)e^{-\varepsilon^{\gamma-1}\hat{x}}\right) + \varepsilon\left(-ebe^{\gamma-1}\hat{x} + (a-eb)\varepsilon^{\gamma-1}\hat{x}e^{-\varepsilon^{\gamma-1}\hat{x}} + A_{L,1} + B_{L,1}e^{-\varepsilon^{\gamma-1}\hat{x}}\right) + O(\varepsilon^2)$$

$$= eb - eb\varepsilon^{\gamma}\hat{x} + \varepsilon A_{L,1} + O(\varepsilon^2)$$

$$y_{out}(x=\varepsilon^{\gamma}\hat{x}) = y_{out,0}(\varepsilon^{\gamma}\hat{x}) + \varepsilon y_{out,1}(\varepsilon^{\gamma}\hat{x}) + O(\varepsilon^2)$$

$$= eb\underbrace{e^{-\varepsilon^{\gamma}\hat{x}}}_{(1-\varepsilon^{\gamma}\hat{x}+O(\varepsilon^{2\gamma}))} + \varepsilon\left(-eb\varepsilon^{\gamma}\hat{x}\left(1-\varepsilon^{\gamma}\hat{x}+O(\varepsilon^{2\gamma})\right) + A_{out,1}\left(1-\varepsilon^{\gamma}\hat{x}+O(\varepsilon^{2\gamma})\right)\right) + O(\varepsilon^2)$$

$\therefore \; y_{out}(x = \varepsilon^{\gamma}\hat{x}) = eb - eb\varepsilon^{\gamma}\hat{x} + \varepsilon A_{out,1} + O(\varepsilon^{1+\gamma}, \varepsilon^{2\gamma}, \varepsilon^{2})$

need $\gamma > 1/2$ to ensure $\varepsilon^{2\gamma}$ term subleading compared to $O(\varepsilon)$ term

Same expansions $\quad A_{L,1} = A_{out,1}$

Note some terms jump order eg. $-eb\varepsilon^{\gamma}\hat{x}$ arises from $y_{out,0}$ even though it's higher order and arises from $y_{L,1}$ in the expansion of the left inner

Matching Right hand boundary layer and outer

- As $\varepsilon \to 0^{+}$ with $\hat{x} < 0$, $\hat{x} = \text{ord}(1)$, $x_R = \varepsilon^{\gamma-1}\hat{x}$

$$
\begin{aligned}
y_R(x_R = \varepsilon^{\gamma-1}\hat{x}) &= y_{R,0}(\varepsilon^{\gamma-1}\hat{x}) + \varepsilon y_{R,1}(\varepsilon^{\gamma-1}\hat{x}) + O(\varepsilon^{2}) \\
&= b + \varepsilon\left(-b\varepsilon^{\gamma-1}\hat{x} + A_{R,1} + B_{R,1}e^{-\varepsilon^{\gamma-1}\hat{x}}\right) + O(\varepsilon^{2}) \\
&= \underbrace{\varepsilon B_{R,1}e^{-\varepsilon^{\gamma-1}\hat{x}}}_{\text{Exponentially leading term}} + b - \varepsilon^{\gamma} b\hat{x} + \varepsilon A_{R,1} + O(\varepsilon^{2})
\end{aligned}
$$