As $\varepsilon \to 0$ with $\hat{x} < 0$, $\hat{x} = \mathrm{ord}(1)$, $x = 1 + \varepsilon^{\gamma}\hat{x}$

$$y_{out}(x = 1+\varepsilon^{\gamma}\hat{x}) = y_{out,0}(1+\varepsilon^{\gamma}\hat{x}) + \varepsilon y_{out,1}(1+\varepsilon^{\gamma}\hat{x}) + O(\varepsilon^2)$$

$$= ebe^{-(1+\varepsilon^{\gamma}\hat{x})} + \varepsilon\left(-eb(1+\varepsilon^{\gamma}\hat{x})e^{-(1+\varepsilon^{\gamma}\hat{x})} + A_{out,1}e^{-(1+\varepsilon^{\gamma}\hat{x})}\right) + O(\varepsilon^2)$$

$$= b(1-\varepsilon^{\gamma}\hat{x} + O(\varepsilon^{2\gamma})) + \varepsilon\left(-b(1+\varepsilon^{\gamma}\hat{x})(1-\varepsilon^{\gamma}\hat{x}+O(\varepsilon^{2\gamma})) + \frac{A_{out,1}}{e}(1-\varepsilon^{\gamma}\hat{x}+O(\varepsilon^{2\gamma}))\right) + O(\varepsilon^2)$$

$$= b - \varepsilon^{\gamma} b\hat{x} - \varepsilon b + \varepsilon A_{out,1}/e + O(\varepsilon^{2\gamma}, \varepsilon^{1+\gamma}, \varepsilon^2)$$

As ~~before~~, $\gamma > 1/2$.

Same expansions : $\boxed{A_{R,1} = A_{out,1}/e - b \; ; \; B_{R,1} = 0}$

Hence

$$\left\{\begin{array}{ll} \underline{\text{BCs}} & A_{L,1} + B_{L,1} = A_{R,1} + B_{R,1} = 0 \\ \underline{\text{Matching}} & A_{L,1} = A_{out,1} \; ; \; B_{R,1} = 0 \; ; \; A_{R,1} = A_{out,1}/e - b \end{array}\right\} \therefore \left\{\begin{array}{l} A_{R,1} = B_{R,1} = 0 \\ A_{out,1} = A_{L,1} = -B_{L,1} \\ \quad = eb \end{array}\right\}$$

Thus
$$y_{L,1}(x_L) = -ebx_L + (a-eb)x_L e^{-x_L} + eb(1-e^{-x_L})$$
$$y_{out,1}(x) = -ebx e^{-x} + ebe^{-x}$$
$$y_{R,1}(x) = -bx_R.$$

Note
$$\lim_{x\to 1} y_{out}(x) = \lim_{x\to 1}\left(ebe^{-x} + \varepsilon eb(1-x)e^{-x} + O(\varepsilon^2)\right) = b + O(\varepsilon^2)$$
$$\lim_{x\to 0} y_{out}(x) = \lim_{x\to 0}\left(ebe^{-x} + \varepsilon eb(1-x)e^{-x} + O(\varepsilon^2)\right) = eb + O(\varepsilon)$$

$\therefore$ $y_{out}(x)$ satisfies BC at $x=1$, at least to $O(\varepsilon^2)$ $\therefore$ Boundary layer not required at $x=1$.

However $\lim_{x\to 0} y_{out}(x) \neq a$ $\therefore$ Boundary layer at $x=0$ required.

Van Dyke's Matching Rule

- Using the intermediate variable $\hat{x}$ yields long calculations
- Van Dyke's matching rule is quicker and usually works :

$$\underbrace{m \text{ terms inner}\left[(n \text{ terms outer})\right]} = \underbrace{n \text{ terms outer}\left[(m \text{ terms inner})\right]}$$

5.14

n terms in the outer expansion, written in terms of the inner variable and expanded to m^{th} order in the inner variable = m terms in the inner expansion written in terms of the outer variable and expanded to n^{th} order in the outer variable

<u>Example</u> At the left hand boundary.

$$y_L(x_L) = A_{L,0} + (a - A_{L,0})e^{-x_L} + O(\varepsilon)$$

$$y_{out}(x) = A_{out,0}\, e^{-x} + O(\varepsilon), \quad x = \varepsilon x_L$$

<u>LHS</u> 1 term outer

$$= A_{out,0}\, e^{-x}$$
$$= A_{out,0}\, e^{-\varepsilon x_L}$$
$$= A_{out,0}\,(1 + O(\varepsilon x_L))$$

<u>RHS</u> 1 term inner

$$= A_{L,0} + (a - A_{L,0})e^{-x_L}$$
$$= A_{L,0} + (a - A_{L,0})e^{-x/\varepsilon} = A_{L,0} + \text{exponentially small}$$

$\therefore \quad A_{out,0} = 1 \text{ term inner}\left[(1 \text{ term outer})\right] = 1 \text{ term outer}\left[(1 \text{ term inner})\right] = A_{L,0}$

$\therefore \quad A_{L,0} = A_{out,0} = eb$

using BC at $x=1$, noting there is no boundary layer there

Note This gives $\lim_{x \to 0} y_{out,0}(x) = \lim_{x_L \to \infty} y_L(x_L)$ as previously observed

Example 2^{nd} order matching

LHS. 2 term outer
$$= A_{out,0}e^{-x} + \varepsilon\left(A_{out,1}e^{-x} - A_{out,0}\,x e^{-x}\right)$$
$$= ebe^{-\varepsilon x_L} + \varepsilon\left(A_{out,1}e^{-\varepsilon x_L} - eb\varepsilon x_L e^{-x_L \varepsilon}\right)$$
$$= eb - \varepsilon eb x_L + \varepsilon A_{out,1} + O(\varepsilon^2)$$

RHS 2 term inner
$$= A_{L,0} + (a - A_{L,0})e^{-x_L} + \varepsilon\left(A_{L,1} - A_{L,1}e^{-x_L} - A_{L,0}x_L + (a - A_{L,0})x_L e^{-x_L}\right)$$
$$= eb + (a-eb)e^{-x/\varepsilon} + \varepsilon\left(A_{L,1} - A_{L,1}e^{-x/\varepsilon} - ebx/\varepsilon + (a-eb)x\,\varepsilon e^{-x/\varepsilon}\right)$$
$$= eb + \varepsilon(A_{L,1}) - ebx + \text{exponentially small terms.}$$

Noting $\varepsilon x_L = x$, we have $A_{L,1} = A_{out,1} = eb$

using BC at $x=1$, noting there is no boundary layer there

$$\therefore \quad y_{out}(x) = eb e^{-x} + \varepsilon eb(1-x)e^{-x} + \dots$$

$$y_L(x_L) = eb + (a-eb)e^{-x_L} + \varepsilon\left(eb\left(1-e^{-x_L}\right) - eb\,x_L + (a-eb)x_L e^{-x_L}\right) + \dots$$

Exercise repeat for 1 term inner[(2 terms outer)] = 2 terms outer[(1 term inner)]

$\boxed{\text{Warning}}$ Treat Logarithmic terms as $O(1)$ in Van Dyke's matching rule due to their size relative to powers.

Composite Expansion

Aim To combine inner and outer expansions to obtain a uniformly valid expansion (for plotting etc)

$$y_{composite} = (p \text{ terms outer}) + (p \text{ terms inner}) - \underbrace{p \text{ terms inner}\left[(p \text{ terms outer})\right]}_{p \text{ terms outer}\left[(p \text{ terms inner})\right] \text{ by Van Dyke.}} \quad p \in \mathbb{N}$$

Subtract p terms inner [(p terms outer)] as it has been counted twice in the overlap region.

Example

p=1 $\quad y_{composite} = y_{out,0}(x) + y_{L,0}(x/\varepsilon)$ − 1 term inner [(1 term outer)]

$$= ebe^{-x} + eb + (a-eb)e^{-x/\varepsilon} - eb$$

$$= ebe^{-x} + (a-eb)e^{-x/\varepsilon}$$

p=2 $\quad y_{composite} = y_{out,0}(x) + \varepsilon y_{out,1}(x) + y_{L,0}(x/\varepsilon) + \varepsilon y_{L,1}(x/\varepsilon)$

− 2 term inner [(2 term outer)]

$$= ebe^{-x} + \varepsilon \cdot eb \cdot (1-x)e^{-x}$$

$$+ eb + (a-eb)e^{-x/\varepsilon} + \varepsilon\left(eb(1-e^{-x/\varepsilon}) - eb\,x/\varepsilon + (a-eb)\frac{x}{\varepsilon}e^{-x/\varepsilon}\right)$$

$$- eb + ebx - \varepsilon eb$$

$$= ebe^{-x} + (a-eb)(1+x)e^{-x/\varepsilon} - \varepsilon eb(1-x)e^{-x} - eebe^{-x/\varepsilon}$$

Choice of rescaling, revisited

In left hand boundary layer, began with scaling $x = \varepsilon^{\alpha} x_L$, $y(x) = y_L(x_L)$.

$$\varepsilon^{1-2\alpha} \frac{d^2 y_L}{dx_L^2} + \varepsilon^{-\alpha} \frac{dy_L}{dx_L} + y_L = 0$$

$\alpha = 0$		Balance	Outer Solution
$0 < \alpha < 1$		Dominant	Overlap region
$\alpha = 1$	Balance		Inner Solution
$\alpha > 1$	Dominant		Sub-inner

The inner and outer solutions can be matched as they share a common term, which is dominant in the overlap region

There are two dominant balances

$$\alpha = 0 \text{ (outer)} \quad \text{and} \quad \alpha = 1 \text{ (inner)}$$

These correspond to distinguished limits in which $x = \text{ord}(1)$ and $x = \text{ord}(\varepsilon)$ respectively.

5.2 Where is the boundary layer?

- For a non-trivial boundary layer, the inner solution decays on approaching the outer region.

Saw this previously in the example

- New example

$$\varepsilon y'' + p(x) y' + q(x) y = 0 \qquad 0 < x < 1$$

$$y(0) = A \quad y(1) = B \qquad 0 < \varepsilon \ll 1$$

$$p, q \text{ smooth; } p(x) > 0$$

RH boundary layer

Let $x = 1 + \delta \hat{x}$ $\quad y(x) = y_R(\hat{x})$ $\quad$ as ever $'$ is derivative wrt argument.

$$\frac{\varepsilon}{\delta^2} y_R'' + \underbrace{p(1+\delta\hat{x})}_{p(1)+O(\delta)} \frac{1}{\delta} y_R' + \underbrace{q(1+\delta\hat{x})}_{q(1)+O(\delta)} y_R = 0$$

Only balance with y_R'' is between 1^{st} & 2^{nd} terms $\therefore$ $\boxed{\varepsilon = \delta}$

$$\therefore \quad y_R'' + \left[p(1) + \varepsilon \hat{x} p'(1) + \cdots\right] y_R' + \varepsilon\left[q(1) + \varepsilon \hat{x} q'(1) + \cdots\right] y_R = 0$$

With $y_R(\hat{x}) \sim y_{R,0} + \varepsilon y_{R,1} + \cdots$

$O(\varepsilon^0)$ $\qquad y_{R,0}'' + p(1) y_{R,0}' = 0$

$$\therefore\ y_{R,0}(\hat{x}) = J + Ke^{-p(1)\hat{x}}$$

Matching $y_{R,0}(-\infty)$ with outer implies $K \equiv 0$, as we have exponential blow up.

$\therefore\ y_{R,0}(\hat{x}) = A$ and no rapid variation in boundary layer

$\therefore$ No boundary layer required.

LH Boundary layer

$$y(x) = y_L(\hat{x}), \quad x = \varepsilon\hat{x}, \qquad y_{L,0} = M + Ne^{-p(0)\hat{x}}$$

Possible to match outer solution without $N \equiv 0$, as $y_{L,0}(\infty)$ finite $\therefore$ Can have boundary layer, illustrating above statement.

Example

$$\varepsilon^2 f'' + 2f(1-f^2) = 0 \qquad |x|<1 \quad f(\pm 1) = \pm 1.$$

- Outer solution one of $f = 0, 1, -1$.
- Near LH boundary $f = -1$ OK; similarly $f = +1$ near RH boundary
- Boundary layer in interior

Let $x = x_0 + \varepsilon X \qquad f(x) = F(X)$

$$\therefore \quad F'' + 2F(1-F^2) = 0 \qquad X \in (-\infty, \infty)$$

$$F \to \pm 1 \qquad \text{as } X \to \pm\infty$$

Solution $\quad F(X) = \tanh(X - X_*)$

$\llcorner$ constant.

Note x_0, X_* undetermined.

By symmetry $f(x) = -f(-x)$ as both satisfy ODE.

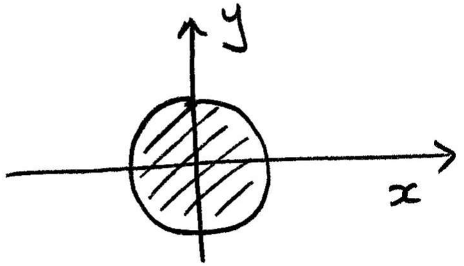
$\therefore f(0) = -f(0) \quad \therefore x_0 = X_* = 0$

$\therefore f = \tanh(x/\varepsilon)$ Agrees with exact solution

Position of transition layer exponentially sensitive to BCs.
Can be analysed with WKBJ method, but beyond scope of course.

5.3 Boundary Layers in PDEs

2D. $\underline{u}.\nabla T = \varepsilon \nabla^2 T$ for $r^2 = x^2+y^2 > 1$ with $T=1$ on $r=1$, and $T \to 0$ as $r \to \infty$,



$$\underline{u} = \nabla \varphi$$

$$\varphi = (r + 1/r)\cos\theta = x + \frac{x}{x^2+y^2}.$$

Outer $T \sim T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \cdots$ as $\varepsilon \to 0^+$ with $r = \text{ord}(1)$.

$O(\varepsilon^0)$ $\underline{u}.\nabla T_0 = 0$, $T_0 \to 0$ as $r \to \infty$, $r > 1$.

On any curve with $\dfrac{d\underline{r}}{ds} = \underline{u}$, $\dfrac{dT_0}{ds} = \nabla T_0 \cdot \dfrac{d\underline{r}}{ds} = \nabla T_0 \cdot \underline{u} = 0$

curve arclength

Also, for $r > 1$ $\dfrac{dx}{ds} = \dfrac{\partial\varphi}{\partial x} = 1 + \dfrac{1}{x^2+y^2} + \dfrac{x.2x.(-1)}{(x^2+y^2)^2} = 1 + \dfrac{y^2-x^2}{(x^2+y^2)^2} = 1 - \dfrac{\cos 2\theta}{r^2} > 0$

$\therefore$ For $r>1$, all such curves go to infinity, where $T_0 = 0$ $\therefore T_0 = 0$ as T_0 invariant on these curves.

<u>Inner</u>

$$\left(1-\tfrac{1}{r^2}\right)\cos\theta\, T_r \;-\; \left(1+\tfrac{1}{r^2}\right)\frac{\sin\theta}{r}\, T_\theta \;=\; \varepsilon\left(T_{rr} + \tfrac{1}{r} T_r + \tfrac{1}{r^2} T_{\theta\theta}\right)$$

Let $r = 1 + \delta(\varepsilon)\rho \qquad T(r,\theta) = T_{inner}(\rho,\theta)$ with $\delta \to 0^+$, $\rho = \text{ord}(1)$, as $\varepsilon \to 0^+$.

$$\therefore \left(1 - \frac{1}{(1+\delta\rho)^2}\right)\frac{\cos\theta}{\delta}\frac{\partial T_{inner}}{\partial\rho} - \left(1 + \frac{1}{(1+\delta\rho)^2}\right)\frac{\sin\theta}{1+\delta\rho}\frac{\partial T_{inner}}{\partial\theta} = \frac{\varepsilon}{\delta^2}\frac{\partial^2 T_{inner}}{\partial\rho^2} + \frac{\varepsilon}{\delta(1+\delta\rho)}\frac{\partial T_{inner}}{\partial\rho} + \frac{\varepsilon}{(1+\delta\rho)^2}\frac{\partial^2 T_{inner}}{\partial\theta^2}$$

$$\therefore \left(2\delta\rho + O(\delta^2)\right)\frac{\cos\theta}{\delta}\frac{\partial T_{inner}}{\partial\rho} - \left(2 + O(\delta)\right)\sin\theta\frac{\partial T_{inner}}{\partial\theta} = \frac{\varepsilon}{\delta^2}\frac{\partial^2 T_{inner}}{\partial\rho^2} \underbrace{+ \frac{\varepsilon}{\delta}\left(1 + O(\delta)\right)\frac{\partial T_{inner}}{\partial\rho} + \varepsilon\left(1 + O(\delta)\right)\frac{\partial^2 T_{inner}}{\partial\theta^2}}_{\text{will never balance}}$$

$\therefore$ Let $\varepsilon/\delta^2 \sim O(1) \quad \therefore$ Let $\delta = \varepsilon^{1/2}$

$$\therefore \quad 2\rho\cos\theta\frac{\partial T_{inner}}{\partial\rho} - 2\sin\theta\frac{\partial T_{inner}}{\partial\theta} = \frac{\partial^2 T_{inner}}{\partial\rho^2}$$

Let $T_{inner} = T_{inner,0} + \varepsilon T_{inner,1} + \dots$

$$2\rho\cos\theta \frac{\partial T_{inner,0}}{\partial \rho} - 2\sin\theta \frac{\partial T_{inner,0}}{\partial \theta} = \frac{\partial^2 T_{inner,0}}{\partial \rho^2}$$

<u>BC</u> $T_{inner,0} = 1$ on $\rho = 0$ (corresponding to $r = 1$) and $T_{inner,0} \to 0$ as $\rho \to \infty$ to match outer.

<u>Seek similiarity solution:</u> $T_{inner,0} = f(\eta)$, $\eta = \rho g(\theta)$.

Then $\frac{\partial T_{inner,0}}{\partial \rho} = g(\theta) f'(\eta)$ $\quad \frac{\partial^2 T_{inner,0}}{\partial \rho^2} = g^2(\theta) f''(\eta)$ $\quad \frac{\partial T_{inner,0}}{\partial \theta} = \rho g'(\theta) f'(\eta)$

Hence $\quad 2\rho\cos\theta \, g(\theta) f'(\eta) - 2\sin\theta \cdot \rho g'(\theta) f'(\eta) = g^2(\theta) f''(\eta)$

$$\therefore \underbrace{\left[\frac{2\cos\theta}{g^2(\theta)} - \frac{2\sin\theta g'(\theta)}{g^3(\theta)}\right] \cdot \rho g(\theta)}_{} \cdot f'(\eta) = f''(\eta)$$

If not negative constant, no solution of this form. Negativity required for f to decay at infinity.

WLOG set constant to be -1 $\therefore$ Solve $2\cos\theta g(\theta) - 2\sin(\theta) g'(\theta) = -g^3(\theta)$

exercise: show this is without loss of generality.

Let $g = 1/p^{1/2}$ converts this into simple ODE and one finds

$$g(\theta) = \frac{|\sin\theta|}{(J + \cos\theta)^{1/2}}$$

J constant.
$J < 1$ blow up
$J > 1$ $g = 0$ at $\theta = \pi$

$\therefore$ If $J > 1$, $T(r,\pi) \sim T_{inner,0}(\rho,\pi) = f(\rho g(\pi)) = f(0) = 1$ By BCs for other angles.

$\therefore$ Upstream heated to $T = 1$, unphysical.

$$\therefore \; J = 1, \quad g(\theta) = \frac{|\sin\theta|}{(1+\cos\theta)^{1/2}}.$$

$\therefore$ We have $f'' + \eta f' = 0 \quad \therefore \; f = Q\int_\eta^\infty e^{-u^2/2}\,du + K$

$T_{inner,0} = f(\eta) \to 0$ as $\rho \to \infty$ i.e. $\eta \to \infty$ $\therefore K = 0$

$T_{inner,0}(\rho = 0) = 1 \;\therefore f(0) = 1 \;\therefore\; f(\eta) = \sqrt{\frac{2}{\pi}}\int_\eta^\infty e^{-u^2/2}\,du$

$\therefore$ Solution to leading order is

$$T(r,\theta) \sim T_{inner,0}(\rho,\theta) = f(\rho g(\theta)) = \sqrt{\frac{2}{\pi}}\int_{\frac{(r-1)}{\varepsilon^{1/2}}\frac{|\sin\theta|}{(1+\cos\theta)^{1/2}}}^{\infty} e^{-u^2/2}\,du$$

solution fails for $\theta \doteq 0$ as we do not satisfy BC at infinity.