Boundary Layer at infinity, logs

$$(x^2y')' + \varepsilon x^2 y y' = 0 \qquad x>1,\ y(1)=0,\ y(\infty)=1$$
$$0<\varepsilon\ll 1$$

Try $y \sim y_0(x) + \varepsilon y_2(x) + \dots$

Know this expansion is incorrect a posteriori (hence the y2) ... to see why, let's try it

$\underline{O(\varepsilon^0)}$ $\quad (x^2 y_0')' = 0 \quad \therefore y_0 = 1 - 1/x$ using boundary conditions.

$\underline{O(\varepsilon^1)}$ $\quad (x^2 y_2')' = -x^2 y_0 y_0' = -1 + 1/x$

$\therefore$ using $y_2(1)=0$, $\quad y_2 = A(1-1/x) - \ln x - \dfrac{\ln x}{x}$

cannot satisfy $y_2(\infty)=0$ (Both $-1+1/x$ are homogeneous solutions to $(x^2 f')'=0$, hence a resonant forcing occurs)

Try $x = X/\delta_1(\varepsilon)$ $\quad y = 1 + \delta_2(\varepsilon) Y(X)$ $\quad$ with $\delta_1, \delta_2 \to 0$, $X = \text{ord}(1)$ as $x\to\infty$

Dominant balance $\qquad \delta_2 \dfrac{d}{dX}\left(X^2 \dfrac{dY}{dX}\right) + \dfrac{\varepsilon\delta_2}{\delta_1} X^2 \dfrac{dY}{dX} + \dfrac{\varepsilon\delta_2^2}{\delta_1} X^2 Y \dfrac{dY}{dX} = 0$

$\delta_1 = \varepsilon$, δ_2 undetermined

small "oh"

Let $Y(X) = Y_0(X) + o(1)$

$$\frac{d}{dX}\left(X^2\frac{dY_0}{dX}\right) + X^2\frac{dY_0}{dX} = 0$$

$$Y_0(X) = B\int_X^\infty \frac{e^{-s}}{s^2}\,ds \qquad \text{nching } Y_0(\infty)=0$$

$\underbrace{\text{Splitting range of integral}}^{\text{exercise}}$, $\quad Y_0(X) = B\left[1/X + \ln X + O(1)\right]$ as $X\to 0^+$.

Need this limit for matching

<u>Intermediate variables</u> $\qquad \hat{x} = \varepsilon^\alpha x = \varepsilon^{\alpha-1} X$

$$y = 1 + \delta_2 Y \sim 1 + \delta_2 B\left[\frac{\varepsilon^{\alpha-1}}{\hat{x}} + \ln(\varepsilon^{1-\alpha}\hat{x}) + \cdots\right] \quad \text{for "inner"}$$

$$y \sim 1 - \frac{\varepsilon^{\alpha}}{\hat{x}} \quad \text{for outer} \qquad \therefore \text{ Let } \delta_2 = \varepsilon,\ B = -1$$

$$\therefore \quad 1 + \delta_2 Y \sim 1 - \frac{\varepsilon^{\alpha}}{\hat{x}} - \varepsilon\ln(\varepsilon^{1-\alpha}\hat{x}) + \cdots$$

$$\sim 1 - \frac{\varepsilon^{\alpha}}{\hat{x}} + \underbrace{(1-\alpha)\varepsilon\ln 1/\varepsilon}_{\text{next term scales with } \varepsilon\ln 1/\varepsilon} - \underbrace{\varepsilon\ln\hat{x}}_{\text{then scale with } \varepsilon}$$

$\therefore$ We should have written $y \sim y_0(x) + \varepsilon\ln 1/\varepsilon\, y_1(x) + \varepsilon y_2(x) + \cdots$

for the outer ...

Now we can match....

$$(x^2 y_1')' = 0 \qquad y_1(x) = C(1 - 1/x) \quad \text{using } y_1(1) = 0.$$

$$\therefore \quad y \sim 1 - \frac{\varepsilon^{\alpha}}{\hat{x}} + \varepsilon\ln\frac{1}{\varepsilon}\, C(1 - \varepsilon^{\alpha}/\hat{x}) + \varepsilon\Big[A(1 - \varepsilon^{\alpha}/\hat{x}) - \ln(\varepsilon^{-\alpha}\hat{x}) - \varepsilon^{\alpha}/\hat{x}\,\ln(\varepsilon^{-\alpha}\hat{x})\Big] + \cdots \quad \text{in intermediate region}$$

for the outer ...

$$\sim 1 - \frac{\varepsilon^{\alpha}}{\hat{x}} + \left(\varepsilon\ln\frac{1}{\varepsilon}\right)\left[C - \alpha\right] + \cdots$$

$$\therefore \quad 1 - \alpha = C - \alpha \quad \text{and} \quad C = 1$$

can now match the inner at leading order

$$\therefore \quad y \sim (1 - 1/x) + \varepsilon\ln 1/\varepsilon\,(1 - 1/x) + O(\varepsilon)$$

5.2.9

Expansion sequence $1, \varepsilon \ln 1/\varepsilon, \varepsilon, \varepsilon^2 \ln 1/\varepsilon, \varepsilon^2 (\ln 1/\varepsilon)^2, \varepsilon^2, \ldots$

Van Dyke rule works only if $(\ln 1/\varepsilon)$ treated as $O(1)$.

but we've used ln(1/epsilon)>>1 in the expansions, so not self-consistent, and thus not satisfactory.

6 Multiple Scales

Van der Pol oscillator

$$\ddot{x} + \varepsilon(x^2-1)\dot{x} + x = 0$$

with $\quad x = 1, \ \dot{x} = 0$ at $t = 0$

Let $x \sim x_0(t) + \varepsilon x_1(t) + \cdots$

With regular perturbation expansion

$$x_0(t) = \cos t$$

$$\ddot{x}_1 + x_1 = (1 - x_0^2)\dot{x}_0 \qquad \text{with} \quad x_1(0) = \dot{x}_1(0) = 0.$$

$$\therefore \quad \ddot{x}_1 + x_1 = (1-\cos^2 t)(-\sin t) = \tfrac{1}{4}\sin 3t - \underbrace{\tfrac{3}{4}\sin t}$$

Will generate resonant terms

$$x_1 = \tfrac{3}{8}(t\cos t - \sin t) - \tfrac{1}{32}(\sin 3t - 3\sin t)$$

$$\therefore \quad x \sim \cos t + \varepsilon\left[\underbrace{\tfrac{3}{8}t\cos t} + \cdots\right] + O(\varepsilon^2)$$

Perturbation expansion breaks down when $t \sim O(1/\varepsilon)$ as x_1 as large as x_0

Long timescales allow corrections to accumulate.

Two timescales

$\tau = t$ - fast timescale of oscillation

$T = \varepsilon t$ - slow timescale of amplitude drift

- Look for a solution of the form

$$x(t,\varepsilon) = x(\tau, T, \varepsilon)$$

treating τ, T as independent.

- $$\frac{d}{dt} = \frac{d\tau}{dt}\frac{\partial}{\partial \tau} + \frac{\partial T}{\partial t}\frac{d}{dT} = \frac{\partial}{\partial \tau} + \varepsilon\frac{\partial}{\partial T}$$

Converting ODE to PDE but freedom in T dependence used to our advantage.

$$\therefore \ddot{x} = x_{tt} = (\partial_\tau + \varepsilon\partial_T)(\partial_\tau + \varepsilon\partial_T)x = x_{\tau\tau} + 2\varepsilon x_{\tau T} + \varepsilon^2 x_{TT}$$

$$\therefore 0 = x_{tt} + \varepsilon(x^2-1)x_t + x$$

$$= x_{\tau\tau} + 2\varepsilon x_{tT} + \varepsilon^2 x_{TT} + \varepsilon(x^2-1)(x_\tau + \varepsilon x_T) + x$$

<u>Expand</u> $\quad x(\tau, T, \varepsilon) = x_0(\tau, T) + \varepsilon x_1(\tau, T) + \ldots$

<u>$O(\varepsilon^0)$</u> $\quad \left\{\begin{array}{l} x_{0\tau\tau} + x_0 = 0 \\ x_0(0) = 1, \; x_{0\tau}(0) = 0 \end{array}\right\}$ $\quad \therefore x_0(\tau, T) = R(T)\cos(\tau + \Theta(T))$

<u>ICS</u> $\quad \underbrace{R(0) = 1, \; \Theta(0) = 0.}_{\text{No other constraints on } R(T),\ \Theta(T) \text{ at this point.}}$

<u>$O(\varepsilon^1)$</u> $\quad x_{1\tau\tau} + x_1 = -x_{0\tau}(x_0^2 - 1) - 2x_{0\tau T}$

$$= \underbrace{2R\Theta_T\cos(\tau+\Theta) + \left(2R_T + \frac{R^3}{4} - R\right)\sin(\tau+\Theta)}_{\text{Will generate resonance}} + \underbrace{R^3/4 \sin 3(\tau+\Theta)}_{\text{will not generate resonance}}$$

with $\quad x_1(0) = 0, \quad x_{1\tau}(0) = -x_{0T}(0) = -R_T(0)$

6.3/

$\therefore$ Let $R(T)\underbrace{\theta_T(T)} = 0 = \left(2R_T + R^3/4 - R\right)$ } "**secular**" conditions – required to avoid resonance

Known as

$\therefore \theta_0 = \text{const}$, with $\theta(0) = 0 \therefore \theta = 0$

$$\frac{dR}{dT} = \frac{1}{2}\left[R - R^3/4\right] \quad \text{with } R(0) = 1 \;\therefore\; R = \frac{2}{(1+3e^{-T})^{1/2}}$$

$$\therefore x(t,\varepsilon) = x(\tau, T, \varepsilon) = \underbrace{\frac{2}{(1+3e^{-\varepsilon t})^{1/2}}}_{\text{Amplitude} \to 2 \text{ as } t\to\infty,\ \varepsilon \text{ fixed.}} \cos t + O(\varepsilon)$$

Higher order

We find $x_1 = -R^3/32 \sin 3\tau + S(T)\sin(\tau + \varphi(T))$

To find $S(T), \varphi(T)$ resonant terms are suppressed for x_2 via secular conditions.

However to suppress resonance we must expand with a slow-slow timescale $T_2 = \varepsilon^2 t$.

To see this, simpler example (do not lecture)

$$\ddot{x} + 2\varepsilon\dot{x} + x = 0$$

$$x = A\underbrace{e^{-\varepsilon t}}_{\text{amplitude drift } t\sim O(1/\varepsilon)} \cos\left(\sqrt{1-\varepsilon^2}\, t + B\right)$$

phase drift on $t \sim O(1/\varepsilon^2)$

$\therefore$ Let $\tau = t,\ T_1 = \varepsilon t,\ T_2 = \varepsilon^2 t$

$$d/dt = \partial/\partial\tau + \varepsilon\,\partial/\partial T_1 + \varepsilon^2\,\partial/\partial T_2$$

} and expand as above

Similarly for higher orders of the Van der Pol oscillator

NB Often presented via a complex representation

eg. Van der Pol

conjugate

$$x_0 = R(T)\cos(\tau + \theta(T)) = \tfrac{1}{2}\left(Ae^{i\tau} + \bar{A}e^{-i\tau}\right)$$

$$A = Re^{i\theta}$$

At $O(\varepsilon^1)$

$$x_{1\tau\tau} + x_1 = -2x_{0\tau T} - (x_0^2 - 1)x_{0\tau}$$

$$= -i\left(A_T e^{i\tau} - \bar{A}_T e^{-i\tau}\right) - \left[\tfrac{1}{4}\left(Ae^{i\tau} + \bar{A}e^{-i\tau}\right)^2 - 1\right] \cdot \frac{i}{2}\left[Ae^{i\tau} - \bar{A}e^{-i\tau}\right]$$

$$= \left[-i\left(A_T - \frac{A(4-|A|^2)}{8}\right)e^{i\tau} + \begin{pmatrix}\text{Complex}\\\text{Conjugate}\end{pmatrix}\right] + \begin{bmatrix}\text{Non}\\\text{secular}\\\text{terms}\end{bmatrix}$$

$\therefore$ Suppressing resonant terms, $e^{\pm i\tau}$

$$A_T = \frac{A}{8}\left(4 - |A|^2\right) \qquad \text{with } A = Re^{i\theta}$$

$$\therefore \quad R_T e^{i\theta} + iR\theta_T e^{i\theta} = \frac{Re^{i\theta}}{8}\left(4 - R^2\right)$$

$$\therefore \quad R_T + iR\theta_T = R/8\,(4 - R^2) \quad \Big\} \text{real \& imag parts} \Big\} \quad \begin{aligned} R\theta_T &= 0 \\ R_T &= R/8\,(4-R^2) \end{aligned}$$

as before

Note sometimes the slow variable, τ, is given the same label as the physical variable t, so that

$$x_0 = R(T)\cos(t + \theta(T)) = \tfrac{1}{2}\left(Ae^{it} + \bar{A}e^{-it}\right) \text{ above etc.}$$

Homogenization

Example $$\frac{d}{dx}\left(D(x, x/\varepsilon)\frac{du}{dx}\right) = f(x) \qquad 0<x<1 \qquad (\dagger)$$

$$u(0)=a, \quad u(1)=b \qquad a,b \in \mathbb{R}^+$$

D, f are smooth, with $0 < D_-(x) < D(x, \overset{X = x/\varepsilon}{X}) < D_+(x)$, with $D_\pm$ continuous.

Question Can $(\dagger)$ be approximated by $\frac{d}{dx}\left(\bar{D}(x)\frac{du}{dx}\right) = f(x)$

$$u(0)=a, \quad u(1)=b$$

for an averaged function $\bar{D}(x)$ ← does not contain fast ε variation.

Multiple Scales Let $u(x,\varepsilon) = \underset{\text{not relabelling as separate variable}}{u(x, X, \varepsilon)}$ with $\underbrace{X = x/\varepsilon}_{\text{fast variable}}$

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial X}$$

$$\therefore \left(\partial_x + \tfrac{1}{\varepsilon}\partial_X\right)\left[D(x,X)\left(\partial_x + \tfrac{1}{\varepsilon}\partial_X\right)u\right] = f(x)$$

$$\therefore \left(\varepsilon\partial_x + \partial_X\right)\left[D(x,X)\left(\varepsilon\partial_x + \partial_X\right)u\right] = \varepsilon^2 f(x)$$

Let $u \sim u_0(x,X) + \varepsilon u_1(x,X) + \cdots$

{ Also assume $u_0, u_1, u_2, \ldots$ bounded for ALL X }

$O(\varepsilon^0)$ $$\left(D(x,X)\, u_{0X}\right)_X = 0$$

$O(\varepsilon^1)$ $$\left(D(x,X)\left[u_{1X} + u_{0x}\right]\right)_X + \left(D(x,X)\, u_{0X}\right)_x = 0$$

$O(\varepsilon^2)$ $$\left(D(x,X)\left[u_{2X} + u_{1x}\right]\right)_X + \left(D(x,X)\left[u_{1X} + u_{0x}\right]\right)_x = f(x)$$

$O(\varepsilon^0)$ $\quad D u_{0X} = c_1(x)$

$$\therefore \quad u_0 = c_2(x) + c_1(x) \underbrace{\int_0^X \frac{ds}{D(x,s)}}_{\text{ord}(X) \text{ as } X\to\infty} \qquad c_1, c_2 \text{ arbitrary}$$

as $\frac{1}{D_+(x)} \leq \frac{1}{D} \leq \frac{1}{D_-(x)}$

u_0 bounded $\quad \therefore \quad u_0 = c_2(x) \quad$ and $c_1(x) = 0$ all x.

$\therefore$ We write $u_0 = u_0(x)$.

$O(\varepsilon^1)$

$$\left(D(x,X)\left[u_{0x} + u_{1X}\right]\right)_X = 0$$

$$D\left[u_{0x} + u_{1X}\right] = d_1(x) \qquad (++)$$

$$\therefore \quad u_1 = d_1(x) \int_0^X \frac{ds}{D(x,s)} - X u_{0x} + d_2(x)$$

Blow up as $X \to \infty$, with both ord(X).

Hence $D_H(x)$ exists.

$$\therefore \quad \text{Let} \quad d_1(x) = \left[\lim_{X\to\infty} \frac{X}{\int_0^X \frac{ds}{D(x,s)}}\right] u_{0x} \equiv D_H(x)\, u_{0x}$$

$O(\varepsilon^2)$

$$\left(D(x,X)\left[u_{2X} + u_{1x}\right]\right)_X = f(x) - d_{1x} \qquad \text{using } (++).$$

$$\therefore \quad D(x,X)\left[u_{2X} + u_{1x}\right] = e_1(x) + \left(f(x) - d_{1x}\right) X$$

$$\therefore \quad u_{2X} = \frac{e_1(x)}{D(x,X)} + \left(f(x) - d_{1x}\right)\frac{X}{D(x,X)} - u_{1x}$$

$$\therefore \quad u_2 = e_1(x) \underbrace{\int_0^X \frac{ds}{D(x,s)}}_{\text{ord}(X) \text{ as } X\to\infty} + \left(f(x) - d_{1x}\right) \underbrace{\int_0^X \frac{s}{D(x,s)}\,ds}_{\text{ord}(X^2) \text{ as } X\to\infty} - \underbrace{\int_0^X u_{1x}\,ds}_{\text{ord}(X) \text{ as } X\to\infty} + e_2(x)$$