NB
$$u_1 = \left\{ \underbrace{\left[\lim_{P\to\infty} \left\{ \frac{P}{\int_0^P \frac{ds}{D(x,s)}} \right\} \int_0^X \frac{ds}{D(s,x)} \right] - X}_{\text{ord}(1) \text{ as } X\to\infty} \right\} u_{0,x} + d_2(x)$$

$$\therefore \int_0^X u_{1x}\, ds = \int_0^X \text{ord}(1)\, u_{0xx} + d_{2x}\, ds = \text{ord}(X) \quad \text{as } X\to\infty$$

$\therefore$ For $u_2(x, X)$ to be bounded, we must have

$$d_{1x} = f(x)$$

$$\therefore \quad \frac{d}{dx}\left(D_H(x)\frac{du_0}{dx}\right) = f(x) \qquad (*)$$

with
$$D_H(x) = \lim_{X\to\infty}\left[\frac{X}{\int_0^X \frac{ds}{D(x,s)}}\right]$$

$$u_0(0) = a, \quad u_0(1) = b$$

$(*)$ Is a "homogenized" ODE

NB If $D(x,s)$ is periodic, say with period 1, D_H simplifies by taking $X \in \mathbb{N}$

$$D_H = \lim_{X\to\infty}\left[\frac{X}{X\int_0^1 \frac{ds}{D(x,s)}}\right] = \frac{1}{\int_0^1 \frac{ds}{D(x,s)}}$$

In higher dimensional problems, periodicity often has to be assumed to make progress

7. WKB Method

- After Wentzel, Kramers, Brillouin (1920s) — Also known as WKBJ where J is for Jeffries. First used by Liouville and Green in [redacted] (1830s).
- Important in semi-classical analysis of quantum mechanics.

Example

$$\left.\begin{array}{l}\varepsilon^2 y'' + y = 0 \\ 0<\varepsilon\ll 1\end{array}\right\} \Rightarrow y = R\cos(x/\varepsilon + \theta) \quad ; R, \theta \in \mathbb{R}, \text{ consts.}$$

High frequency oscillations. What if frequency of oscillation depends on the slow scale...

Example

$$\varepsilon^2 y'' + q(x) y = 0 \qquad q(x) > 0$$

$$0<\varepsilon\ll 1$$

Try multiple scales $\quad x = \varepsilon X \quad \therefore \quad \frac{d^2 y}{dX^2} + q(\varepsilon X) y = 0$

treat x, X as independent

$$\frac{dy}{dX} = \frac{\partial y}{\partial X} + \frac{\partial x}{\partial X}\frac{\partial y}{\partial x} = \left(\frac{\partial}{\partial X} + \varepsilon\frac{\partial}{\partial x}\right) y$$

$$\therefore \quad y_{XX} + 2\varepsilon y_{xX} + \varepsilon^2 y_{xx} + q(x) y = 0$$

Let $y = y_0(x, X) + \varepsilon y_1(x, X) + \cdots$

<u>0th order</u> $\quad y_{0XX} + q(x)\, y_0 = 0 \quad \therefore \quad y_0 = R(x)\cos\left(\sqrt{q(x)}\,X + \theta(x)\right)$

<u>1st order</u> $\quad y_{1XX} + q(x)\, y_1 = -2 y_{0xX}$

$$= -2\left[R(x)\cos\left(\sqrt{q(x)}\,X + \theta(x)\right)\right]_{xX}$$

$$= +2\left[\sqrt{q(x)}\,R(x)\sin\left(\sqrt{q(x)}\,X + \theta(x)\right)\right]_x$$

$$= 2\frac{\partial}{\partial x}\left[\sqrt{q(x)}\,R(x)\right]\sin\left(\sqrt{q(x)}\,X+\theta(x)\right) + 2\frac{\partial}{\partial x}\left(\sqrt{q(x)}\,X+\theta(x)\right)\sqrt{q}\,R(x)\cos\left(\sqrt{q(x)}\,X+\theta(x)\right)$$

Both terms on RHS are resonant.

<u>Secular conditions</u> : $\quad \frac{\partial}{\partial x}\left(\sqrt{q(x)}\,R(x)\right) = 0 = \sqrt{q}\,R\,\frac{\partial}{\partial x}\left(\sqrt{q(x)}\,X + \theta(x)\right)$

$q>0 \therefore$ Either $R(x)=0$ (trivial solution obvious and useless)

or $\quad \frac{\partial}{\partial x}\left(\sqrt{q(x)}\,X + \theta(x)\right) = 0 \quad$ i.e. $\quad \frac{\partial}{\partial x}\left(\sqrt{q}\right)X + \theta'(x) = 0$

i.e. $\quad X = -\dfrac{\theta'(x)}{\partial/\partial x\left(\sqrt{q}\right)} \quad$ ✱ function of x cannot be equal to X <u>for all X.</u>

Happens whenever frequency of fast oscillation drifts on a slowscale.

$\therefore$ <u>Try a WKB expansion</u> $\quad y = \exp\left[\frac{i}{\varepsilon}\varphi(x)\right]A(x,\varepsilon)$

$$y' = e^{i\varphi/\varepsilon}\left[i\varphi' A/_{\varepsilon} + A' \right]$$

$$y'' = e^{i\varphi/\varepsilon}\left[\frac{i\varphi'}{\varepsilon}\left(i\varphi' A/_{\varepsilon} + A'\right) + \left(i\varphi' A/_{\varepsilon} + A'\right)' \right]$$

$$\therefore \quad \varepsilon^2 e^{i\varphi/\varepsilon}\left[\frac{-\varphi'^2 A}{\varepsilon^2} + \frac{2i\varphi' A' + i\varphi'' A}{\varepsilon} + A'' \right] + q e^{i\varphi/\varepsilon} A = 0$$

$$\therefore \quad \varepsilon^2 A'' + \{2i\varepsilon\varphi' A'\} + \{-\varphi'^2 + i\varepsilon\varphi'' + q\} A = 0$$

Let $A = A_0 + \varepsilon A_1 + \cdots$

$O(\varepsilon^0)$ $\quad \{-\varphi'^2 + q\} A_0 = 0 \quad \therefore$ For $A_0 \not\equiv 0$, $\quad \underline{\underline{\varphi'^2 = q}}$

$O(\varepsilon^1)$ $\quad 2i\varphi' A_0' + i\varphi'' A_0 + \underbrace{\{-\varphi'^2 + q\}}_{0} A_1 = 0$

$$\therefore \quad \frac{2A_0'}{A_0} + \frac{\varphi''}{\varphi'} = 0$$

$$\therefore \quad \log A_0^2 \varphi' = \text{Const} \quad \therefore \quad A_0 = \frac{\alpha_0}{(\varphi')^{1/2}} \qquad \alpha_0 \in \mathbb{C}$$

$O(\varepsilon^{n+1})$ $\quad A_{n-1}'' + 2i\varphi' A_n' + i\varphi'' A_n = 0$

$$\therefore \quad \left((\varphi')^{1/2} A_n\right)' = -\frac{1}{2i(\varphi')^{1/2}} A_{n-1}''$$

$$\therefore \quad A_n = \frac{i}{2(\varphi')^{1/2}} \int^{x} \frac{A_{n-1}''(s)}{2(\varphi')^{1/2}(s)}\, ds$$

At leading order

$$y \sim \frac{\alpha_+}{(q(x))^{1/4}} \exp\left[\frac{i}{\varepsilon}\int^x \sqrt{q(s)}\, ds\right] + \frac{\alpha_-}{(q(x))^{1/4}} \exp\left[-\frac{i}{\varepsilon}\int^x \sqrt{q(s)}\, ds\right]$$

In principle can go to higher orders as A_n generally known.

$\alpha_\pm \in \mathbb{C}$.

Method breaks down near $\varphi' = 0$, as amplitude blows up.

↑ Fix at turning points considered later.

Example

Find eigenvalues with $\lambda \gg 1$ for $p(x)$ a positive function and

$$y'' + \lambda p(x) y = 0 \qquad 0 < x < 1 \qquad y(0) = 0 \quad y(1) = 0$$

Let $\lambda = 1/\varepsilon^2$, $0 < \varepsilon \ll 1$. Then

$$\varepsilon^2 y'' + p(x) y = 0$$

WKB Let $y = e^{i\varphi/\varepsilon} A(x,\varepsilon) \sim e^{i\varphi/\varepsilon} \sum_{n=0}^{\infty} \varepsilon^n A_n(x)$.

$O(\varepsilon^0)$ $\quad \varphi'^2 = p \quad \therefore \varphi' = \pm\sqrt{p(x)} \quad \therefore \varphi = \pm \int_0^x p(s)^{1/2} ds$

const of integration absorbed into A_0.

$O(\varepsilon^1)$ $\quad 2\varphi' A_0' + \varphi'' A_0 = 0 \quad \therefore A_0 = \dfrac{\text{const}}{(p(x))^{1/4}}$

Two lin. independent solutions

$$y_+ \sim A_0 e^{i\varphi/\varepsilon} \qquad y_- \sim A_0 e^{-i\varphi/\varepsilon}$$

General solution, at leading orders:

$$y \sim \alpha A_0(x) \cos(\varphi(x)/\varepsilon) + \beta A_0(x) \sin\left(\frac{\varphi(x)}{\varepsilon}\right)$$

$$\alpha, \beta \in \mathbb{R}.$$

$y(0) = 0 \quad \therefore \alpha = 0$

$y(1) = 0$ satisfied at leading order only if $\beta A_0(1) \sin\left(\frac{\varphi(1)}{\varepsilon}\right) = o(1)$ (small "oh")

We have $A_0(1) \neq 0$, $\beta > 0$ for a non-trivial solution

$$\therefore \quad \varphi(1) \sim n\pi\varepsilon \quad \text{as } \varepsilon \to 0.$$

$$\therefore \frac{1}{\sqrt{\lambda_n}} = \varepsilon_n \sim \frac{\varphi(1)}{n\pi} = \frac{1}{n\pi}\int_0^1 \sqrt{p(x)}\,dx$$

λ_n: n^{th} eigenvalue

$$\therefore \lambda_n \sim \left(\frac{n\pi}{\int_0^1 \sqrt{p(x)}\,dx} \right)^2 \quad \text{as } n \to \infty$$

Example **Semi-Classical Quantum. Turning Points.**

The non-dimensional steady state Schrödinger equation for the even wave-functions of the simple harmonic oscillator is given by

$$\psi'' - x^2\psi = -E\psi$$
$$\psi \to 0 \text{ as } x \to \infty, \quad \psi'(0) = 0.$$

Find the large, $E \gg 1$, energy eigenvalues.

Let $y = \psi$. $x = \bar{x}/\sqrt{\varepsilon}$ with $\varepsilon = 1/E$. Then, dropping bars,

$$\varepsilon^2 y'' + (1 - x^2) y = 0$$
$$y(\infty) = 0, \; y'(0) = 0, \quad 0 < \varepsilon \ll 1.$$

Let $y = e^{i\varphi/\varepsilon} A(x, \varepsilon) \sim e^{i\varphi/\varepsilon} \sum_{n=0}^{\infty} \varepsilon^n A_n(x)$

WKB $O(\varepsilon^0)$ $\varphi' = \pm\sqrt{1-x^2}$ $O(\varepsilon^1)$ $A_0 = \dfrac{\text{const}}{(1-x^2)^{1/4}}$

Hence

For $0 < x < 1$, $$y \sim \frac{M_0}{(1-x^2)^{1/4}} e^{1/\varepsilon \int_0^x \sqrt{1-s^2}\, ds} + \frac{N_0}{(1-x^2)^{1/4}} e^{-1/\varepsilon \int_0^x \sqrt{1-s^2}\, ds}$$

using $y'(0) = 0$

$$\sim \frac{P_0}{(1-x^2)^{1/4}} \cos\left(\frac{1}{\varepsilon}\int_0^x \sqrt{1-s^2}\, ds\right)$$

For $x > 1$ $$y \sim \frac{Q_0}{(x^2-1)^{1/4}} e^{-1/\varepsilon \int_1^x \sqrt{s^2-1}\, ds}$$ using $y(\infty) = 0$

However, these breakdown near $x \simeq 1$ as $\varphi'(1) = 0$.

Resolve using matched asymptotics

Inner region around $x = 1$ Let $x = 1 + \delta_1(\varepsilon) X$ $Y(X) = \delta_2(\varepsilon)\, y(x)$

$$\frac{\varepsilon^2}{\delta_1^2} \frac{d^2Y}{dX^2} + \Big(1 - \underbrace{(1 + 2\delta_1 X + \delta_1^2 X^2)\Big) Y}_{2\delta_1 XY + \delta_1^2 X^2 Y} = 0$$

Dominant balance when $2\delta_1^3 = \varepsilon^2$ $\therefore$ Let $\delta_1 = \frac{\varepsilon^{2/3}}{2^{1/3}}$.

δ_2 undetermined as yet

With $Y = Y_0(X) + o(1)$ ← small "oh"

$$\frac{d^2 Y_0}{dX^2} - XY_0 = 0 \quad \therefore \quad Y_0 = R_0 \, Ai(X) + S_0 \, Bi(X)$$ where Ai, Bi are Airy functions.

Airy Functions

$$Ai(X) = \frac{1}{\pi} \int_0^\infty \cos\left(t^3/3 + Xt\right) dt \sim \frac{1}{2\sqrt{\pi}\, X^{1/4}} e^{-2/3 X^{3/2}} \quad \text{as } X \to \infty$$

$$\sim \frac{1}{\sqrt{\pi}\,(-X)^{1/4}} \sin\left(2/3\,(-X)^{3/2} + \pi/4\right) \quad \text{as } X \to -\infty.$$

$$Bi(X) = \frac{1}{\pi} \int_0^\infty \exp\left(-t^3/3 + Xt\right) dt \sim \frac{1}{\sqrt{\pi}\, X^{1/4}} e^{2/3 X^{3/2}} \quad \text{as } X \to \infty$$

$$\sim \frac{1}{\sqrt{\pi}\,(-X)^{1/4}} \cos\left(2/3\,(-X)^{3/2} + \pi/4\right) \quad \text{as } X \to -\infty$$

Matching Inner ($X \to \infty$) with RH outer ($x \to 1^+$)

$S_0 = 0$ else Y_0 blows up as $X \to \infty$.

On matching everything scales with $\frac{1}{X^{1/4}} e^{-2/3 X^{3/2}}$ whether using Van Dyke or intermediate region. Naively one gets simply $0=0$. Thus, on matching, insist the coefficients in front of $\frac{1}{X^{1/4}} e^{-2/3 X^{3/2}}$ match.

<u>Matching</u> (intermediate variable)

Let $x-1 = \delta_1^\beta \hat{x} = \delta_1 X \qquad (0<\beta<1)$ with $\hat{x} = \text{ord}(1)$, $x \to 1$, $X \to \infty$, $\hat{x} > 0$.

$$Y_0 = R_0 \, Ai\left(\frac{\hat{x}}{\delta_1^{1-\beta}}\right) \sim \frac{R_0}{2\sqrt{\pi}} \frac{(\delta_1^{1-\beta})^{1/4}}{\hat{x}^{1/4}} \exp\left[-2/3 \cdot \frac{1}{(\delta_1^{1-\beta})^{3/2}} \hat{x}^{3/2}\right]$$

$$y \sim \frac{Q_0}{[(x-1)(x+1)]^{1/4}} \exp\left[-1/\varepsilon \int_1^x \sqrt{s^2-1}\, ds\right]$$

$s^2-1 = (s-1)(s+1)$, $s = 1+\eta$

$$\int_1^x \sqrt{s^2-1}\, ds = \int_0^{x-1} \eta^{1/2} 2^{1/2} \sqrt{1+\eta/2}\, d\eta$$

$$= \sqrt{2} \cdot 2/3 (x-1)^{3/2} + \cdots$$

$$= \frac{2\sqrt{2}}{3} \delta_1^{3\beta/2} \hat{x}^{3/2} + \ldots$$

$$\therefore 1/\varepsilon \int_1^x \sqrt{s^2-1}\, ds = \frac{1}{(2^{1/3}\delta_1)^{3/2}} \frac{2\sqrt{2}}{3} \delta_1^{3\beta/2} \hat{x}^{3/2} + \ldots$$

$$\therefore\ y \sim \frac{Q_0}{2^{1/4}\delta_1^{\beta/4}\hat{x}^{1/4}} \exp\left[-\frac{2}{3}\frac{1}{(\delta_1^{1-\beta})^{3/2}}\hat{x}^{3/2}\right] + \cdots$$

$$\therefore\ Y = \delta_2 y \sim \frac{Q_0\,\delta_2(\varepsilon)}{2^{1/4}(\delta_1)^{\beta/4}\hat{x}^{1/4}} \exp\left[-\tfrac{2}{3}\frac{1}{(\delta_1^{1-\beta})^{3/2}}\hat{x}^{3/2}\right] + \cdots \sim \frac{R_0\,\delta_1^{1/4}}{2\sqrt{\pi}}\frac{1}{\delta_1^{\beta/4}}\frac{1}{\hat{x}^{1/4}} \exp\left[-\frac{2}{3}\frac{\hat{x}^{3/2}}{(\delta_1^{1-\beta})^{3/2}}\right]$$

$$\therefore\ \delta_2 = \delta_1^{1/4} = \left(\varepsilon^{2/3}/2^{1/3}\right)^{1/4} = \frac{1}{2^{1/12}}\varepsilon^{1/6} \quad \text{and} \quad Q_0 = \frac{1}{2^{3/4}\sqrt{\pi}}R_0$$

<u>Matching</u> inner ($X \to -\infty$) with LH outer ($x \to 1^-$).

Let $x - 1 = \delta_1^{\gamma}\hat{x} = \delta_1 X \quad (0<\gamma<1)$ with $\hat{x} = \text{ord}(1)$, $x \to 1$, $X \to -\infty$, $\hat{x} < 0$.

$$Y_0 = R_0\,Ai\left(\frac{\hat{x}}{\delta_1^{1-\gamma}}\right) \sim \frac{R_0(\delta_1)^{\frac{1-\gamma}{4}}}{\sqrt{\pi}(-\hat{x})^{1/4}} \sin\left(\frac{2}{3}(-\hat{x})^{3/2}\frac{1}{(\delta_1^{1-\gamma})^{3/2}} + \frac{\pi}{4}\right)$$

$$y \sim \frac{P_0}{2^{1/4}(-\hat{x})^{1/4}\delta_1^{\gamma/4}} \cos\left(\pi/4\varepsilon - \frac{1}{\varepsilon}\int_x^1 \sqrt{1-s^2}\,ds\right) \quad \text{using } \int_0^1 \sqrt{1-s^2}\,ds = \pi/4$$

$$\therefore\ y \sim \frac{P_0}{2^{1/4}(-\hat{x})^{1/4}\delta_1^{\gamma/4}} \cos\left(\frac{\pi}{4\varepsilon} - \frac{1}{\varepsilon}\cdot\frac{2\sqrt{2}}{3}(1-x)^{3/2} + \dots\right)$$

← Substituting $s = 1-\eta$ in integral and using $\sqrt{1-s^2} = \eta^{1/2}(2^{1/2} + O(\eta))$

$$\sim \frac{P_0}{2^{1/4}(-\hat{x})^{1/4}\delta_1^{\gamma/4}} \cos\left(\pi/4\varepsilon - \underbrace{\frac{2\sqrt{2}}{3\varepsilon}\delta_1^{3\gamma/2}}_{\frac{2}{3}\frac{1}{(\delta_1^{1-\gamma})^{3/2}}}(-\hat{x})^{3/2} + \dots\right)$$

$$\sim \delta_2^{-1} Y_0 = \frac{R_0}{\sqrt{\pi}\,(-\hat{x})^{1/4}\delta_1^{\gamma/4}} \sin\left(\pi/4 + 2/3(-\hat{x})^{3/2}\frac{1}{(\delta_1^{1-\gamma})^{3/2}}\right)$$

With $w = 2/3(-\hat{x})^{3/2}\dfrac{1}{(\delta_1^{1-\gamma})^{3/2}}$,

$$\frac{P_0}{2^{1/4}}\cos\left(\pi/4\varepsilon - w\right) \sim \frac{R_0}{\sqrt{\pi}}\sin\left(\pi/4 + w\right)$$

$$\therefore\ \frac{P_0}{2^{1/4}}\left[\cos\frac{\pi}{4\varepsilon}\cos w + \sin\left(\pi/4\varepsilon\right)\sin w\right] \sim \frac{R_0}{\sqrt{\pi}}\left[\sin\pi/4\cos w + \cos\pi/4\sin w\right]$$

$$\therefore\ \frac{P_0}{2^{1/4}}\cos\frac{\pi}{4\varepsilon} \sim \frac{R_0\sin\pi/4}{\sqrt{\pi}}\ , \quad \frac{P_0\sin\left(\pi/4\varepsilon\right)}{2^{1/4}} \sim \frac{R_0}{\sqrt{\pi}}\cos\pi/4$$

For $P_0, R_0 \neq 0$ $\qquad \tan\left(\frac{\pi}{4\varepsilon}\right) \sim \cot(\pi/4) = 1$ as $\varepsilon \to 0$

$$\therefore \quad \frac{\pi}{4\varepsilon} \sim \frac{\pi}{4} + n\pi \qquad \text{as } n \to \infty, \quad \text{with } n \in \mathbb{N}$$

$$\therefore \quad E_n = \frac{1}{\varepsilon_n} = 1 + 4n \qquad \text{as } n \to \infty, \text{ for the energy levels.}$$

<u>Once this holds</u> $\qquad \cos\left(\frac{\pi}{4\varepsilon}\right) \sim \cos(\pi/4 + n\pi) = \frac{1}{\sqrt{2}}(-1)^n \qquad \therefore \; P_0 = \frac{2^{1/4}}{\sqrt{\pi}}(-1)^n R_0 = 2\,(-1)^n Q_0$ } Connection formula

$$y_n \sim \frac{Q_0}{(x^2-1)^{1/4}} e^{-1/\varepsilon_n \int_1^x \sqrt{s^2-1}\, ds} \qquad x > 1, \; x \not\approx 1$$

$$\sim \frac{2^{1/12}}{\varepsilon^{1/6}} \cdot 2^{3/4} \cdot \sqrt{\pi}\, Q_0 \,\mathrm{Ai}\left(\frac{2^{1/3}(x-1)}{\varepsilon_n^{2/3}}\right) \qquad x \simeq 1$$

$$\sim \frac{2(-1)^n Q_0}{(1-x^2)^{1/4}} \cos\left(\frac{1}{\varepsilon_n}\int_0^x \sqrt{1-s^2}\, ds\right) \qquad x<1,\; x \not\approx 1, \quad \varepsilon_n = \frac{1}{1+4n}, \quad n \gg 1.$$