Solid Mechanics

Lecture Notes

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Contents

HEALTH WARNING: The following lecture notes are meant as a rough guide to the lectures. They are not meant to replace the lectures. You should expect that some material in these notes will not be covered in class and that extra material will be covered during the lectures (especially longer proofs, examples, and applications). Nevertheless, I will try to follow the notation and the overall structure of the notes as much as possible.

WHAT IS EXAMINABLE? The official examinable content of the course is described on the web page `https://courses.maths.ox.ac.uk/`.

I will not discuss any further the possible content of the examination paper, in class or otherwise. I am happy to answer questions during and after the lecture (or by emails) regarding the material. But, any question or email referring to examinable or non-examinable content of the course shall be ignored.

Useful formulas

A few key formulas and definitions that we will be using throughout this course.

Tensor calculus Here ϕ, $\mathbf{v}$, $\mathbf{T}$ are, respectively, scalar, vector and 2^{nd}-order tensor fields defined on a moving body. Upper case refer to the reference configuration, lower case to the current configuration.

$\mathbf{F} = \mathrm{Grad}\,\mathbf{x} = \frac{\partial x_i}{\partial X_j}\mathbf{e}_i \otimes \mathbf{E}_j$	Deformation Gradient	$(T1)$
$J = \det\mathbf{F}$	Determinant of $\mathbf{F}$	$(T2)$
$\mathrm{grad}\,\mathbf{v} = \frac{\partial \mathbf{v}}{\partial x_i} \otimes \mathbf{e}_i$	Definition of the gradient of a vector	$(T3)$
$\mathrm{grad}\,\mathbf{T} = \frac{\partial \mathbf{T}}{\partial x_i} \otimes \mathbf{e}_i$	Definition of the gradient of a tensor	$(T4)$
$\mathrm{div}\,\mathbf{T} = \frac{\partial T_{ij}}{\partial x_i}\mathbf{e}_j$	Definition of the divergence of a tensor	$(T5)$
$\mathrm{Grad}\,\phi = \mathbf{F}^{\mathbf{T}}\mathrm{grad}\,\phi$	Gradients of a scalar	$(T6)$
$\mathrm{Grad}\,\mathbf{v} = (\mathrm{grad}\,\mathbf{v})\mathbf{F}$	Gradients of a vector	$(T7)$
$\mathrm{Div}\,\mathbf{v} = J\,\mathrm{div}\,(J^{-1}\mathbf{F}\mathbf{v})$	Divergences of a vector	$(T8)$
$\mathrm{Div}\,\mathbf{T} = J\,\mathrm{div}\,(J^{-1}\mathbf{F}\mathbf{T})$	Divergences of a tensor	$(T9)$
$\mathrm{div}(J^{-1}\mathbf{F}) = 0$	An important identity	$(T10)$
$\frac{\partial}{\partial\lambda}(\det\mathbf{T}) = (\det\mathbf{T})\mathrm{tr}\left(\mathbf{T}^{-1}\frac{\partial \mathbf{T}}{\partial\lambda}\right)$	A useful identity. λ is a scalar	$(T11)$

Kinematics

$\mathbf{F} = \mathrm{Grad}\ \mathbf{x}(\mathbf{X}, t)$	The deformation gradient	$(K1)$
$J = \det\mathbf{F}$	Determinant of $\mathbf{F}$	$(K2)$
$\mathrm{d}\mathbf{x} = \mathbf{F}\mathrm{d}\mathbf{X}$	Transformation of line element	$(K3)$
$\mathrm{d}\mathbf{a} = J\mathbf{F}^{\text{-T}}\mathrm{d}\mathbf{A}$	Transformation of area element	$(K4)$
$\mathrm{d}v = J\mathrm{d}V$	Transformation of volume element	$(K5)$
$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$	Right Cauchy-Green tensor	$(K6)$
$\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$	Left Cauchy-Green tensor	$(K7)$
$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{1}\right)$	Euler strain tensor	$(K8)$
$\mathbf{L} = \mathrm{grad}\ \mathbf{v}$	Velocity gradient	$(K9)$
$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}$	Evolution of the deformation gradient($\mathbf{v}$: velocity)	$(K10)$
$\dot{J} = J\mathrm{div}\ \mathbf{v}$	Evolution of the volume element	$(K11)$
$\mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^{\mathrm{T}}\right)$	Eulerian strain rate tensor	$(K12)$
$\mathbf{W} = \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^{\mathrm{T}}\right)$	Rate of rotation tensor	$(K13)$

Mechanics Here ρ is the mass density, T, the Cauchy stress tensor, $\mathbf{v}$ the velocity, $W = J\Psi$, where Ψ is the internal energy density.

$\dot{\rho} + \rho\mathrm{div}\mathbf{v} = 0$	Conservation of mass (Eulerian form)	$(M1)$
$\mathrm{div}\mathbf{T} + \rho\mathbf{b} = \rho\dot{\mathbf{v}}$	Conservation of linear momentum (Eulerian form)	$(M2)$
$\mathrm{div}\mathbf{T}^{\mathrm{T}} = \mathbf{T} = 0$	Conservation of angular momentum (Eulerian form)	$(M3)$
$\dot{W} = J\mathrm{tr}(\mathbf{T}\mathbf{D})$	Conservation of energy (Eulerian form)	$(M4)$
$\mathbf{T}\mathbf{n} = \mathbf{t}$	Surface traction associated with $\mathbf{T}$($\mathbf{n}$: normal outward unit)	$(M5)$

Material frame indifference Consider two different frame, connected by a rigid body motion $\mathbf{x}^* = \mathbf{Qx} + \mathbf{c}$. Let ϕ, $\mathbf{v}$, $\mathbf{T}$ be, respectively, scalar, vector and 2^{nd}-order tensor fields, and $\mathbf{F}$ the deformation gradient. Then

$$\begin{array}{lll} & \mathbf{F}^* = \mathbf{QF} & \\ \phi & \text{is objective if} & \phi^* = \phi \\ \mathbf{v} & \text{is objective if} & \mathbf{v}^* = \mathbf{Qv} \\ \mathbf{T} & \text{is objective if} & \mathbf{T}^* = \mathbf{QTQ}^{\mathrm{t}} \end{array}$$

List of assumptions

- **Continuum assumption.** We consider a body with reference configuration $\mathcal{B}_0 \subset \mathbb{R}^3$. At time t, the body occupies the current configuration $\mathcal{B}_t \subset \mathbb{R}^3$. A material point, initially at $\mathbf{X} \in \mathcal{B}_0$ is mapped to a point $\mathbf{x} \in \mathcal{B}_t$ by the one-parameter mapping $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ so that $\boldsymbol{\chi} : \mathcal{B}_0 \to \mathcal{B}_t$. The continuum assumption states that $\boldsymbol{\chi}$ is a bijection mapping for all time t. This implies that we can write $\mathbf{x} = \boldsymbol{\chi}^{-1}(\mathbf{X})$. We further assume that this mapping is twice continuously differentiable in $\mathbf{X}$ and t. This assumption can be relaxed in problems involving phase boundaries (with possible jumps in the first derivative). In many instances and applications, we will assume that $\boldsymbol{\chi}$ is actually smooth.

- **Conservation of mass.** We assume the existence of a scalar density function $\rho = \rho(\mathbf{x})$ defined on the body $\mathcal{B}_t$ and whose integral over any material subset $\Omega_t \subset \mathcal{B}_t$ of the body remains constant in time. So that $\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} \rho(\mathbf{x}) \mathrm{d}v = 0$

- **Balance of linear momentum.** We assume that the rate of change of linear momentum of an arbitrary material subset $\Omega_t \subset \mathcal{B}_t$ is equal to sum of all the forces acting on Ω_t.

- **Balance of angular momentum.** We assume that the rate of change of angular momentum of an arbitrary material subset $\Omega_t \subset \mathcal{B}_t$ with respect to a given point is equal to sum of all the torques acting on Ω_t with respect to the same point.

- **Polar media.** For polar media, we assume that the body is not subject to body or contact torques.

- **Cauchy's postulate.** Cauchy's postulate simply state that the traction vector on a given surface element depends smoothly on the the normal to that element.

Foreword

The traditional approach to continuum mechanics is to consider the regime of small deformations, where a material is slightly perturbed from an unstressed configuration. In that regime, the governing equations and constitutive relationships are linear. An example of this approach is found in This *theory of linear elasticity* that has been successfully developed over the last two centuries to address many fundamental problems of physics and engineering [93].

Starting in the 1940's, it was found that the theory of linear elasticity was inadequate to model the response of elastomers such as rubbers in large deformations [128, 147]. Similarly, many tissues and organs also operate in large deformations. For instance, large arteries in mammals are typically stretched between 20% to 60% from their unloaded configurations [68, 83] and their response to loads is drastically different from the response of elastomers.

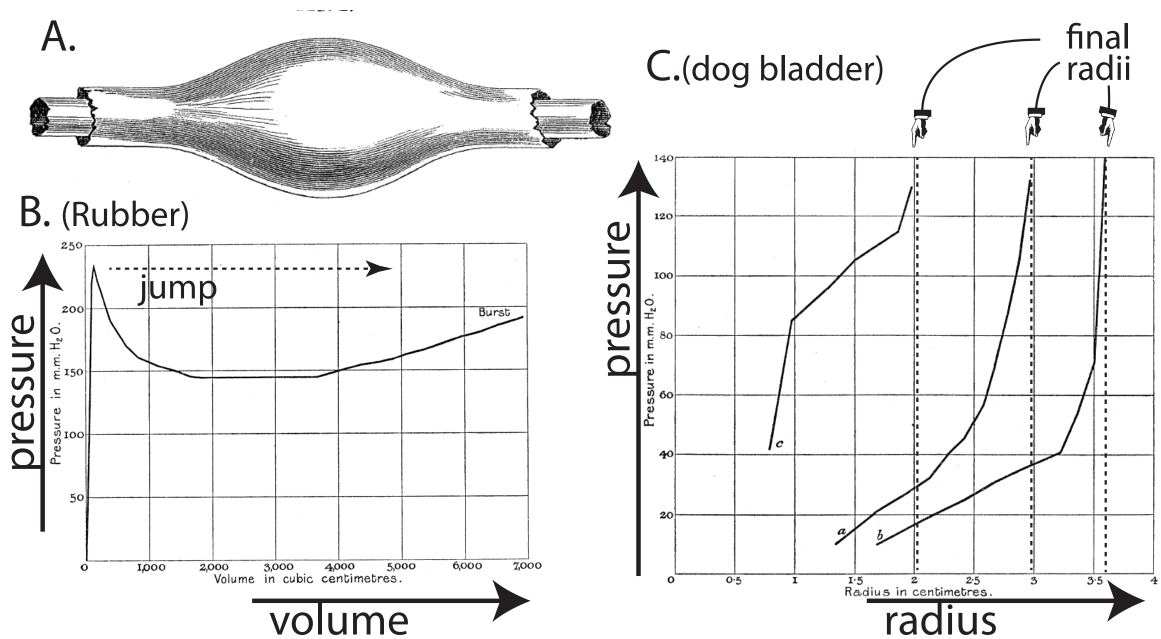


Figure 1: The importance of nonlinearities and large deformations is demonstrated in a pressure experiment as originally investigated by Mallock in 1890 [102] (A). The experiment consists in increasing the pressure inside a tube or a sphere while recording the radius of the bulge. B. In the case of rubber, the experiment of Osborne and Sutherland [122] shows a sudden limit-point instability at a critical pressure. Past that pressure, the radius suddenly increases. C. In a similar experiment performed on a dog bladder, the behavior of the system is qualitatively different and the instability disappears. In this case, it will be increasingly harder to increase the size of the bulge by increasing the pressure.

A striking example of the difference between rubbers and soft tissues is observed in experiments first performed in the nineteenth Century [102, 122]. In these experiments, a cylindrical elastic membrane made out of rubber or soft tissue is pressurized (see Figure 1). At a critical pressure, the radius of the rubber cylinder will suddenly jump to a new equilibrium, whereas for soft tissues, the radius will evolve continuously to an asymptotic radius. Therefore, a continuum theory for the mechanical response of materials in large deformations requires the general *theory of nonlinear elasticity*, which, by contrast to the theory of linear elasticity, assumes neither small deformations, a particular choice of constitutive law, nor a particular symmetry of the material.

In these note, we review the general theory of nonlinear solid mechanics, starting with the description of kinematics and moving to the Cauchy equations governing the response of a continuum. The first part is a general approach to continuum mechanics. The second part is the specialisation to elasticity. In the theory of elasticity, these equations are complemented by constitutive laws relating stresses to strains. The theory is presented at an introductory level and further details can be found in the textbooks of Ogden [118], Gurtin, Fried, and Anand [60, 61], Truesdell and Noll [146], or Antman [7].

1 Introduction: one-dimensional elasticity

■ **Overview** We explore elasticity in one dimension to give a general idea of the different steps necessary to develop a general theory of elasticity.

1.1 A one-dimensional theory

Here, we consider a one-dimensional continuum that can only deform along its length. Therefore, there is no bending, twisting, or shearing, just stretching. The emphasis here is on understanding the different steps that enter in the development of a full theory of continuum in the simplest possible context. The steps are

1) **Kinematics:** A description of the possible deformations. The definition of *strains*, given by geometry. In our context, it is just the stretch along the line.

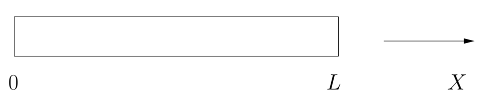
2) **Mechanics:** The definitions of *stresses* and *forces* acting on the medium. Then a statement of balance laws based on the balance of linear and angular momenta, this is applicable to all continuum media but for our problem, linear momentum is sufficient.

3) **Constitutive laws:** A statement of the relationship between stresses and strains. This is where we describe the response of the material under loads.

The results of these three steps is a closed set of equations whose solutions (with appropriate boundary conditions and initial data) is a description of the stresses and deformations in a particular body under a particular set of forces.

1.2 Kinematics

Consider a 1D continuum of length L. Any material point is labelled by $X \in [0, L]$. The motion or deformation is the mapping $x = x(X, t)$, which is assumed smooth and invertible to ensure that there is no material separation, discontinuity, or overlap. The kinematics is



fully described by a the stretch and the velocity at one point.

$$\lambda = \frac{\partial x}{\partial X}, \quad \text{stretch}; \qquad \dot{x} = V(X,t) = \frac{\partial x}{\partial t}, \quad \text{velocity}. \tag{1}$$

Since the mapping is invertible, we have $\lambda > 0$ for all motion. The identity mapping $x = X$ corresponds to the stress-free (Langrangian) configuration.

Motion: The velocity of a material point is $V(X, t) = \dot{x} = \partial x/\partial t$. Since $X = X(x, t)$ is invertible, we can write,

$$v(x,t) = \dot{x}(X(x,t),t), \tag{2}$$

where v is the velocity at the spatial point x.

The acceleration of a point is,

$$\ddot{x}(X,t) = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}, \qquad \text{or} \qquad a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}, \tag{3}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x}, \tag{4}$$

is the *material time derivative.*

1.3 Dynamics

We use two fundamental principles to obtain equations for the motion of a continuum: the conservation of mass and the balance of linear momentum (in a general theory we will also need the balance of angular momentum but it does not play a role in 1D).

1.3.1 Conservation of mass

We define ρ to be the linear density in the current configuration (mass per unit length as measured in the current configuration) and ρ_0 the linear density in the reference configuration. Assuming no mass is created, we have

$$\int_{X_1}^{X_2} \rho_0 \,\mathrm{d}x = \int_{x_1}^{x_2} \rho \,\mathrm{d}x, \tag{5}$$

with $x_1 = x(X_1, t), x_2 = x(X_2, t)$. Since $\mathrm{d}x = \lambda \mathrm{d}X$, we have

$$\int_{X_1}^{X_2} \rho_0 \,\mathrm{d}X = \int_{X_1}^{X_2} \rho\lambda \,\mathrm{d}X, \tag{6}$$

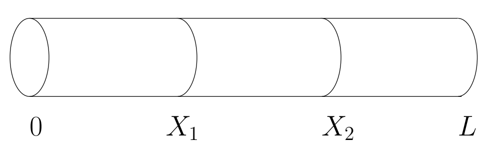
which implies that $\lambda\rho = \rho_0$, the Lagrangian conservation of mass. This is the first conservation law.

1.3.2 Balance of linear momentum

The general principle for the balance of linear momentum is

$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{linear momentum}) = \text{force acting on the system.}$$

Let is decompose this into the following pieces:



1) **The linear momentum**:

$$\int_{X_1}^{X_2} \rho_0 \dot{x} \,\mathrm{d}X \tag{7}$$

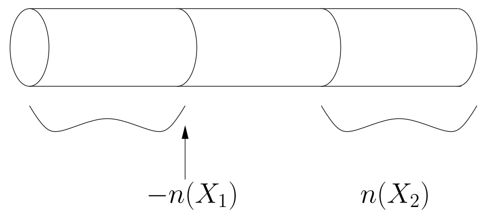
2) **forces**: themselves further decompose into forces due to **external (body)** forces or **internal (contact)** forces:

- **Body forces**,

$$\int_{X_1}^{X_2} \rho_0 f \, \mathrm{d}X \tag{8}$$

where$f$ is the density of body force (force per unit mass).

- **Contact forces:** force the material exerts on itself.



This material exerts a force $n(X_2)$ on $[0, X_2]$ counted positive (tensile) if the force is in the direction of the axis, compressive otherwise. Therefore, from the principle of action=reaction, the contact force acting on the segment $[X_1, X_2]$ is $n(X_2)-n(X_1)$.

Therefore, the balance of linear momentum for a one-dimensional continuum implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{X_1}^{X_2} \rho_0 \dot{x} \, \mathrm{d}X = \int_{X_1}^{X_2} \rho_0 f \, \mathrm{d}X + n(X_2) - n(X_1) \tag{9}$$

We can obtain an expression with a single integral by moving the derivative inside the integral and using the fundamental theorem of calculus,

$$\int_{X_1}^{X_2} \frac{\partial n}{\partial X} \, \mathrm{d}X = n(X_2) - n(X_1). \tag{10}$$

That is

$$\int_{X_1}^{X_2} \left(\rho_0 \ddot{x} \, \mathrm{d}X - \rho_0 f \frac{\partial n}{\partial X} \right) \mathrm{d}X = 0. \tag{11}$$

This relation is valid $\forall\ X_1,\ X_2$, so that, we can localise the integral (assuming continuity of the integrand) to obtain

$$\rho_0 a = \rho_0 f + \frac{\partial n}{\partial X}. \tag{12}$$

This is an equation for the force $n(X)$ in the material (Cauchy first equation). This equation is in the reference configuration (all quantities depend on the material variable X and time t). We can obtain an equation in the current configuration by using $\mathrm{d}X = \lambda^{-1}\,\mathrm{d}x$

$$\rho a = \rho f + \frac{\partial n}{\partial x}. \tag{13}$$

But what is $\partial n / \partial x$? We need a constitutive law to close the system.

1.3.3 Constitutive laws

To close the problem, we need to relate the stresses to the strains, that is a relationship between σ and λ such as Hooke's law

$$n = K(\lambda - 1). \tag{14}$$

This Hookean law is only typically valid for small deformations. For large deformations, we will assume in general that the material is *hyperelastic*, that is the constitutive law derives from a potential Ψ capturing the elastic energy associated with deformation so that

$$n = f(\lambda) = \frac{\partial \Psi}{\partial \lambda}. \tag{15}$$

with the requirement that $f(1) = 0$ and that the derivative of f at $\lambda = 1$ exists. For such systems, the Hooke constant $K = f'(1)$ is then simply the linearised behaviour for small deformations around the stress-free state. The theory of three-dimensional elasticity developed in next section when applied to the uniaxial extension of an incompressible rectangular *neo-Hookean* bar suggests the following nonlinear law

$$n = \frac{K}{3}(\lambda^2 - \lambda^{-1}), \tag{16}$$

Close to $\lambda = 1$, we recover Hooke's law (as shown in Fig 2). More generally, materials

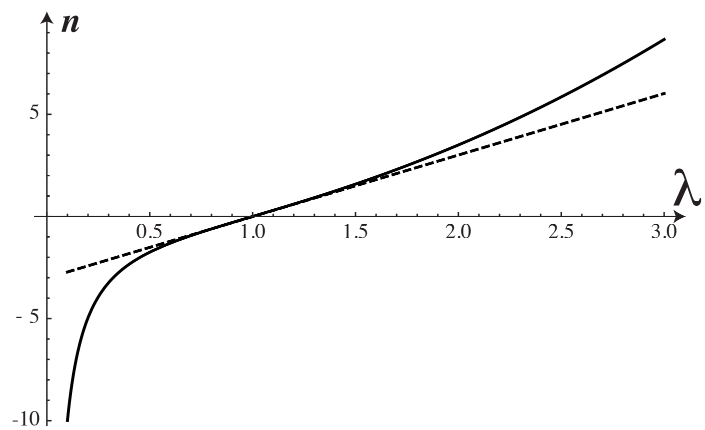


Figure 2: Comparison between the linear (dash) and nonlinear (solid) Hookean response for $K = 3$.

that show strain-stiffening (increase in stiffness for large deformations) or strain-softening (decrease in stiffness) can be modelled by various functions of the stretch.

2 Kinematics

■ **Overview** We develop a completely general theory for the deformation of three-dimensional bodies with no assumptions on displacements. To do so, we introduce two configurations and relate them through the motion in time and the deformation gradient. The deformation gradient is naturally defines as a two-point tensors and its analysis require some tensor calculus.

Our first task consists in defining properly the deformation of a body in a three-dimensional space. To do so, we first define our body as occupying a region of a three-dimensional space. After a deformation, the body also occupies a region in space and the motion of each point of the body, from its initial to its current positions, can be defined by a mapping. The relative motion of nearby points can be extracted from this mapping and its derivatives in space and time. It allows us to define key quantities such as strains and stretch. Therefore, we first introduce basic notion of kinematics in three dimensions.

A *body* B is a set of material points whose elements can be put into a 1-1 correspondence (bijection) with points in a region $\mathcal{B} \subset \mathbb{E}^3$. As the body moves or deforms, this set can change with time $t \in \mathbb{R}$. In nonlinear elasticity, one considers multiple configurations for the description of a body and denotes by $\mathcal{B}_t$ (or $\mathcal{B}$, when there is no possibility of confusion) the *configuration* of B at time t.

For static systems, we use $\mathcal{B}_0$ as the *initial configuration*, typically an unloaded configuration, and $\mathcal{B}$ for the *current configuration* where loads are applied. The initial configuration, $\mathcal{B}_0$, is parameterized by material points relative to the position vector $\mathbf{X}_0$ with origin $\mathbf{O}$ and the current configuration, $\mathcal{B}$, by the position vector $\mathbf{x}$ with origin $\mathbf{o}$.

The basic assumption for the deformation of a continuum is that the body retains its integrity and that material points do not overlap during a deformation. Therefore, both $\mathcal{B}_0$ and $\mathcal{B}$ are bijections of B, and there exists an invertible mapping, pictured in Figure 3, called *deformation* or *motion* $\boldsymbol{\chi} : \mathcal{B}_0 \to \mathcal{B}_t$ such that

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t), \quad \forall\ \mathbf{X} \in \mathcal{B}_0 \qquad \text{and} \qquad \mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}, t), \quad \forall\ \mathbf{x} \in \mathcal{B}_t. \tag{17}$$

In the absence of phase boundaries, singularities, or jumps, we assume that this mapping is twice continuously differentiable in space and smooth in time.

It is convenient to use two orthonormal rectangular Cartesian bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ to represent vectors in the initial and current configuration

$$\mathbf{x} = x_i \mathbf{e}_i, \qquad \mathbf{X} = X_i \mathbf{E}_i, \tag{18}$$

where summation over repeated indices is always assumed unless explicitly specified.

The convention in continuum mechanics is to refer to coordinates in the current configuration, or quantities expressed with these coordinates, as *Eulerian* or *spatial*. Coordinates in the initial configuration, or quantities expressed with these coordinates, are referred to as *Lagrangian*, *referential*, or *material*.

2.1 Scalars, vectors, and tensors

To describe the deformations of a body B, we attach, at each material point, physical quantities known as *fields*, which make continuum mechanics a theory of fields. These quantities can be scalar fields, such as density, temperature; vector fields, such as velocity, acceleration, force; or tensor fields, such as deformation gradients or stress and strain tensors. These different fields can all be understood as tensor fields of different orders. By definition, a scalar

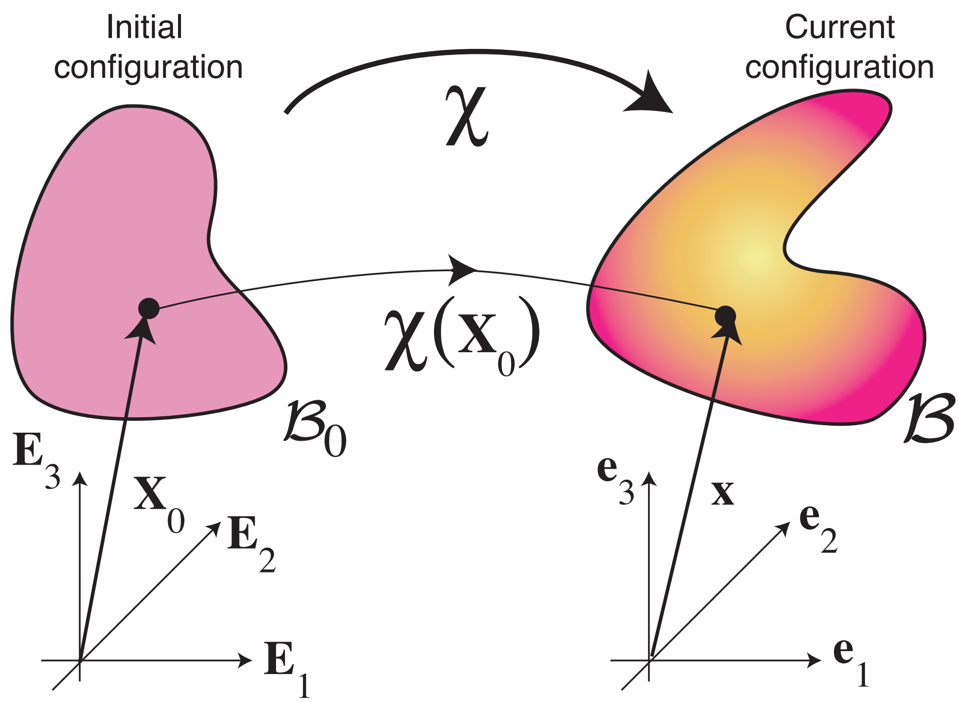


Figure 3: Basic kinematic of nonlinear elasticity. Two configurations are defined. The deformation is a 1-1 map between points of the reference configuration $\mathcal{B}_0$ and points of the current configuration $\mathcal{B}$.

field is a tensor field of order 0 and a vector field is a first-order tensor field. Higher-order tensor fields require the definition of the tensor product.

The *scalar product* between two vectors $\mathbf{a} = a_i\mathbf{e}_i$ and $\mathbf{b} = b_i\mathbf{e}_i$, in the same vector space, follows the usual definition:

$$\mathbf{u} \cdot \mathbf{v} = u_i v_i, \tag{19}$$

and is used to defined the *Euclidean norm*

$$|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}. \tag{20}$$

We can also use the scalar product to define the tensor product. Consider two vectors $\mathbf{u} = u_i\mathbf{e}_i$ and $\mathbf{v} = v_i\mathbf{E}_i$, not necessarily defined in the same vector space. Then, the *tensor product* , $\mathbf{u}\otimes\mathbf{v}$, of these two vectors is a second-order tensor such that, for an arbitrary vector $\mathbf{a} = a_i\mathbf{E}_i$,

$$(\mathbf{u} \otimes \mathbf{v})\mathbf{a} = (\mathbf{v} \cdot \mathbf{a})\mathbf{u}. \tag{21}$$

This definition implies that the vector $\mathbf{v}$ and $\mathbf{a}$ must belong to the same vector space, but in general, $\mathbf{u}$ can belong to a different space. We see from this definition that a second-order tensor maps vectors from one vector space to vectors in another vector space.

Explicitly, the tensor product is

$$\mathbf{u} \otimes \mathbf{v} = u_i\mathbf{e}_i \otimes v_j\mathbf{E}_j = u_i v_j \mathbf{e}_i \otimes \mathbf{E}_j. \tag{22}$$

When there is no possibility of confusion, we can write the *components* of the second-order tensor $\mathbf{u} \otimes \mathbf{v}$ in the Cartesian bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ as $(\mathbf{u} \otimes \mathbf{v})_{ij} = u_i v_j,\ i, j = 1, 2, 3$.

Equipped with the tensor product, we can define a general *second-order tensor* in the Cartesian bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ as

$$\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{E}_j \quad \Longleftrightarrow \quad T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{E}_j, \tag{23}$$

which implies that for a vector $\mathbf{a} = a_j\mathbf{E}_j$,

$$(\mathbf{Ta})_j = T_{ij}a_j. \tag{24}$$

We define the *matrix of components* of a tensor in Cartesian coordinates by $[\mathbf{T}]$ such that $[\mathbf{T}]_{ij} = T_{ij}$. We can then extend most definitions and identities of traditional linear algebra to tensors.

A particularly important class of second-order tensor are the tensors whose component matrices are square matrices. For these second-order tensors, the *determinant* and *trace* of a second-order tensor are defined, respectively, as

$$\det \mathbf{T} = \det([\mathbf{T}]), \qquad \text{tr } \mathbf{T} = \text{tr }([\mathbf{T}]) = T_{ii}. \tag{25}$$

In particular, in three dimensions, we have

$$\det \mathbf{T} = \det([\mathbf{T}]) = \epsilon_{ijk}T_{1i}T_{2j}T_{3k}, \tag{26}$$

where ϵ_{ijk} denotes the usual *Levi-Civita permutation symbols* or *permutation symbols*, that is $\epsilon_{ijk} = 1$ is (i, j, k) for an even permutation of $(1, 2, 3)$, -1 if it is an odd permutation, and 0 if any index is repeated.

Similarly, the matrix of the transpose of a tensor is the transpose of the matrix, that is

$$[\mathbf{T}^\mathsf{T}] = [\mathbf{T}]^\mathsf{T} \tag{27}$$

and a tensor is *symmetric*, $\mathbf{T}^\mathsf{T} = \mathbf{T}$, if and only if $T_{ij} = T_{ji}$.

The product of two tensors $\mathbf{S}$ and $\mathbf{T}$ is only defined when the image of a vector by $\mathbf{T}$ is in the domain of $\mathbf{S}$. Then, for an arbitrary vector $\mathbf{a}$, we have

$$(\mathbf{ST})\mathbf{v} = \mathbf{S}(\mathbf{Ta}). \tag{28}$$

In such cases, the matrix of the product is the product of the two matrices:

$$[\mathbf{ST}] = [\mathbf{S}][\mathbf{T}]. \tag{29}$$

A tensor $\mathbf{S}$ is an *orthogonal tensor* if

$$\mathbf{SS}^\mathsf{T} = \mathbf{S}^\mathsf{T}\mathbf{S} = \mathbf{1}, \tag{30}$$

where $\mathbf{1}$ is the identity tensor defined as $(\mathbf{1})\mathbf{a} = \mathbf{a}\ \forall\mathbf{a}$. As expected, it follows that the components of an orthogonal tensor is an orthogonal matrix. The group of all orthogonal tensors in three dimensions is denoted $O(3)$. A *proper orthogonal tensor* is an orthogonal tensor with the additional property $\det \mathbf{S} = 1$. The group of all proper orthogonal tensors in three dimensions is denoted $SO(3)$. Orthogonal and proper orthogonal tensors are particularly useful to characterize rotations and proper rotations in a three-dimensional space.

We can also contract two tensors together to obtain a scalar by introducing the double contraction

$$\mathbf{S} : \mathbf{T} = \text{tr}(\mathbf{ST}) = S_{ij}T_{ji}. \tag{31}$$

If the determinant of a tensor $\mathbf{T}$ does not vanish, the matrix of *inverse* of $\mathbf{T}$ is the inverse of the matrix:

$$[\mathbf{T}^{-1}] = [\mathbf{T}]^{-1}. \tag{32}$$

Explicitly, for a tensor $\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{E}_j$, we have

$$\mathbf{T}^{-1} = ([\mathbf{T}]^{-1})_{ij}\mathbf{E}_i \otimes \mathbf{e}_j, \tag{33}$$

so that

$$\mathbf{T}\mathbf{T}^{-1} = \mathbf{1} = \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \tag{34}$$

$$\mathbf{T}^{-1}\mathbf{T} = \mathbf{1} = \delta_{ij}\mathbf{E}_i \otimes \mathbf{E}_j, \tag{35}$$

where δ_{ij} is the usual Kronecker delta's symbol ($\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$).

The tensor product can also be used to define higher-order tensors. For instance, a third-order tensor, $\mathbf{Q}$, and a fourth-order tensor $\mathcal{Q}$, in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are defined as

$$\mathbf{Q} = Q_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k, \tag{36}$$

$$\mathcal{Q} = Q_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l. \tag{37}$$

However, in this case, the equivalence with linear algebra is lost and identities for higher-order tensors must be obtained by following the rules defining the tensor products and playing the game of indices. Note that higher-order tensors can also be defined with respect to multiple vector spaces.

2.2 Spatial derivatives of tensors

Next, we define spatial derivatives of scalar, vector, and tensor fields. We have two sets of spatial variables, the Lagrangian variables $\mathbf{X}$ and the Eulerian variables $\mathbf{x}$. We can therefore define different types of spatial derivatives depending on the description of a given quantity.

We first consider the case where ϕ, u, $\mathbf{T}$ are scalar, vector and tensor fields respectively over $\mathbf{x}$, that is

$$\phi = \phi(\mathbf{x}, t), \qquad \mathbf{u} = u_i(\mathbf{x}, t)\mathbf{e}_i, \qquad \mathbf{T} = T_{ij}(\mathbf{x}, t)\mathbf{e}_i \otimes \mathbf{e}_j. \tag{38}$$

We define the Eulerian *gradient* of scalar and vector functions as

$$\text{grad}\ \phi = \frac{\partial \phi}{\partial \mathbf{x}} = \frac{\partial \phi}{\partial x_i}\mathbf{e}_i, \tag{39}$$

$$\text{grad}\ \mathbf{u} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \frac{\partial \mathbf{u}}{\partial x_j} \otimes \mathbf{e}_j = \frac{\partial (u_i\mathbf{e}_i)}{\partial x_j} \otimes \mathbf{e}_j = \frac{\partial u_i}{\partial x_j}\mathbf{e}_i \otimes \mathbf{e}_j. \tag{40}$$

The gradient is an operation that increases the order of the tensor and is defined, in general, as the operation

$$\text{grad}(\,\cdot\,) = \frac{\partial (\,\cdot\,)}{\partial x_j} \otimes \mathbf{e}_j. \tag{41}$$

It follows from this definition that

$$\text{grad}(\phi\mathbf{u}) = \mathbf{u} \otimes \text{grad}\,\phi + \phi\ \text{grad}\,\mathbf{u}. \tag{42}$$

Similarly, we define the gradient of a second-order tensor as

$$\text{grad}\ \mathbf{T} = \frac{\partial}{\partial x_k}\left(T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j\right) \otimes \mathbf{e}_k = \frac{\partial T_{ij}}{\partial x_k}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k. \tag{43}$$

The *divergence* decreases the order of a tensor by contracting indices. For a vector, we have simply

$$\text{div } \mathbf{u} = \frac{\partial u_i}{\partial x_i}. \tag{44}$$

For a second-order tensor, the contraction can take place on the first or second index depending on the convention. Here, we choose to define the divergence as a contraction on the first index, that is

$$\text{div } \mathbf{T} = \frac{\partial T_{ij}}{\partial x_k}\mathbf{e}_j\,(\mathbf{e}_i \cdot \mathbf{e}_k) = \frac{\partial T_{ij}}{\partial x_i}\mathbf{e}_j. \tag{45}$$

With this particular definition of the divergence operator, the *divergence theorem*, applied on a domain $\Omega \subset \mathbb{R}^3$, reads

$$\int_{\partial\Omega} \mathbf{T}\mathbf{n}\,\mathrm{d}a = \int_{\Omega} \text{div}\,(\mathbf{T}^{\mathsf{T}})\,\mathrm{d}v. \tag{46}$$

We consider now spatial derivatives with respect to Lagrangian coordinates, that is Φ, U, and $\mathbf{T}$ are now fields over $\mathbf{X}$:

$$\Phi = \phi(\mathbf{X}, t), \qquad \mathbf{U} = u_i(\mathbf{X}, t)\mathbf{E}_i, \qquad \mathbf{T} = T_{ij}(\mathbf{X}, t)\mathbf{e}_i \otimes \mathbf{E}_j. \tag{47}$$

The Lagrangian gradient is then the operation

$$\text{Grad}(\,\cdot\,) = \frac{\partial(\,\cdot\,)}{\partial X_j} \otimes \mathbf{E}_j. \tag{48}$$

Note that we use the lower case "grad" and "div" to describe spatial derivatives with respect to Eulerian coordinates and the upper case "Grad" and "Div" for spatial derivatives with respect to Lagrangian coordinates. Explicitly, we have

$$\text{Grad } \Phi = \frac{\partial \Phi}{\partial \mathbf{X}} = \frac{\partial \Phi}{\partial X_i}\mathbf{E}_i, \tag{49}$$

$$\text{Grad } \mathbf{U} = \frac{\partial \mathbf{U}}{\partial \mathbf{X}} = \frac{\partial \mathbf{U}}{\partial X_j} \otimes \mathbf{E}_j = \frac{\partial (U_i \mathbf{E}_i)}{\partial X_j} \otimes \mathbf{E}_j = \frac{\partial U_i}{\partial X_j}\mathbf{E}_i \otimes \mathbf{E}_j, \tag{50}$$

$$\text{Grad } \mathbf{T} = \frac{\partial}{\partial X_k}\,(T_{ij}\mathbf{e}_i \otimes \mathbf{E}_j) \otimes \mathbf{E}_k = \frac{\partial T_{ij}}{\partial X_k}\mathbf{e}_i \otimes \mathbf{E}_j \otimes \mathbf{E}_k. \tag{51}$$

The divergence is then

$$\text{Div } \mathbf{U} = \frac{\partial U_i}{\partial X_i}, \tag{52}$$

$$\text{Div } \mathbf{T} = \frac{\partial T_{ij}}{\partial X_k}\mathbf{e}_j\,(\mathbf{E}_i \cdot \mathbf{E}_k) = \frac{\partial T_{ij}}{\partial X_i}\mathbf{E}_j. \tag{53}$$

Note that these definitions can be appropriately modified for the case where $\mathbf{T} = T_{ij}\mathbf{E}_i \otimes \mathbf{E}_j$ by changing $\mathbf{e}_j \to \mathbf{E}_j$ in the definition of Grad and Div.

2.3 Derivatives in curvilinear coordinates

It is often convenient to describe a body and a deformation with respect to curvilinear coordinates. For instance, it is natural to use cylindrical coordinates to describe simple deformations of a cylinder. We use the curvilinear coordinates $\{q_1, q_2, q_3\}$ in the current configuration and $\{Q_1, Q_2, Q_3\}$ in the reference configuration. These coordinates are related to the Cartesian coordinates in each configuration through the position vectors $\mathbf{x} = \mathbf{x}(q_1, q_2, q_3)$

and $\mathbf{X} = \mathbf{X}(Q_1, Q_2, Q_3)$. Here, we use greek subscripts to denote quantities defined in non-Cartesian coordinates. For instance, we associate to each coordinate set, a set of basis vectors

$$\mathbf{e}_\alpha = h_\alpha^{-1} \frac{\partial \mathbf{x}}{\partial q_\alpha}, \quad \mathbf{E}_\alpha = H_\alpha^{-1} \frac{\partial \mathbf{X}}{\partial Q_\alpha}, \qquad \alpha = 1, 2, 3, \tag{54}$$

where h_α and H_α are *scale factors*, used to normalize the basis vectors:

$$h_\alpha = \left| \frac{\partial \mathbf{x}}{\partial q_\alpha} \right|, \quad H_\alpha = \left| \frac{\partial \mathbf{X}}{\partial Q_\alpha} \right|, \qquad \alpha = 1, 2, 3, \tag{55}$$

For brevity, we restrict our attention to a set of orthogonal coordinate, so that

$$\mathbf{e}_\alpha \cdot \mathbf{e}_\beta = \delta_{\alpha\beta}, \qquad \mathbf{E}_\alpha \cdot \mathbf{E}_\beta = \delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2, 3. \tag{56}$$

We define the gradient, $\operatorname{grad} \mathbf{T} = \nabla \otimes \mathbf{T}$, of a tensor $\mathbf{T}$ at a point $\mathbf{x} \in \mathcal{B}$ as the tensor that maps a vector $\mathbf{v}$ in the tangent space of $\mathcal{B}$ at $\mathbf{x}$ onto the infinitesimal variation of $\mathbf{T}$ along a path going through $\mathbf{x}$ with tangent $\mathbf{v}$. For any given $\mathbf{v}$, we define a path Γ, parameterized by its arc length s, going through $\mathbf{x}$ and tangent to $\mathbf{v}$. The operation of the gradient on a vector $\mathbf{v}$ is

$$\begin{aligned}
(\nabla \otimes \mathbf{T})\mathbf{v} &= \frac{\mathrm{d}\mathbf{T}(\Gamma(s))}{\mathrm{d}s} = \lim_{\mathrm{d}s \to 0} \frac{\mathbf{T}(\Gamma(s + \mathrm{d}s)) - \mathbf{T}(\Gamma(s))}{\mathrm{d}s} \\
&= \frac{\partial \mathbf{T}(\mathbf{x})}{\partial x^\alpha} \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} = \frac{\partial \mathbf{T}(\mathbf{x})}{\partial x^\alpha} \delta_{\alpha\beta} \frac{\mathrm{d}q_\beta}{\mathrm{d}s} \\
&= \frac{\partial \mathbf{T}}{\partial q_\alpha} (h_\alpha^{-1} \mathbf{e}_\alpha \cdot h_\beta \mathbf{e}_\beta) \frac{\mathrm{d}q_\beta}{\mathrm{d}s} \\
&= \left(\frac{\partial \mathbf{T}}{\partial q_\alpha} \otimes h_\alpha^{-1} \mathbf{e}_\alpha \right) (\mathbf{v}),
\end{aligned} \tag{57}$$

where we have used the fact that the tangent to Γ at $\mathbf{x}$ is $h_\beta \mathbf{e}_\beta (\mathrm{d}q_\beta / \mathrm{d}s)$. Since this operation applies to arbitrary vectors $\mathbf{v}$, the *gradient of a tensor in orthogonal curvilinear coordinates* is

$$\operatorname{grad} \mathbf{T} = h_\alpha^{-1} \frac{\partial \mathbf{T}}{\partial q_\alpha} \otimes \mathbf{e}_\alpha, \qquad \operatorname{Grad} \mathbf{T} = H_\alpha^{-1} \frac{\partial \mathbf{T}}{\partial Q_\alpha} \otimes \mathbf{E}_\alpha. \tag{58}$$

Similarly, we define the *divergence of a tensor field* $\mathbf{T}$ as $\operatorname{div} \mathbf{T} = \nabla \cdot \mathbf{T}$, that is

$$\operatorname{div} \mathbf{T} = h_\alpha^{-1} \mathbf{e}_\alpha \cdot \frac{\partial \mathbf{T}}{\partial q_\alpha}, \qquad \operatorname{Div} \mathbf{T} = H_\alpha^{-1} \mathbf{E}_\alpha \cdot \frac{\partial \mathbf{T}}{\partial Q_\alpha}. \tag{59}$$

Note that we take the scalar product on the left which corresponds to the contraction on the first index of $\mathbf{T}$. Choosing Cartesian coordinates $\{q_1, q_2, q_3\} = \{x_1, x_2, x_3\}$ leads to $h_\alpha = 1 \, \forall \alpha$, and the definitions (58) and (59) reduce to those of the previous section.

As an example, consider the choice of polar coordinates $\{q_1, q_2\} = \{r, \theta\}$ in the Euclidean plane. The position vector is $\mathbf{x} = r \cos\theta \mathbf{e}_1 + r \sin\theta \mathbf{e}_2$, so that

$$\begin{aligned}
\mathbf{e}_r &= \frac{\partial \mathbf{x}}{\partial r}, \qquad h_r = 1, \\
\mathbf{e}_\theta &= \frac{1}{r} \frac{\partial \mathbf{x}}{\partial \theta}, \quad h_\theta = r.
\end{aligned} \tag{60}$$

Hence, according to (58), the gradient of a scalar ϕ is

$$\operatorname{grad} \phi = (\partial_r \phi) \, \mathbf{e}_r + \frac{1}{r} (\partial_\theta \phi) \, \mathbf{e}_\theta, \tag{61}$$

and we recover the usual formula of vector calculus. Similarly, if we write a second-order tensor $\mathbf{T}$ in polar representation:

$$\mathbf{T} = T_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + T_{r\theta}\mathbf{e}_r \otimes \mathbf{e}_\theta + T_{\theta r}\mathbf{e}_\theta \otimes \mathbf{e}_r + T_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta, \tag{62}$$

then its divergence is the first order tensor

$$\begin{aligned} \operatorname{div}\mathbf{T} &= \left(\frac{1}{r}\frac{\partial}{\partial r}(rT_{rr}) + \frac{1}{r}\frac{\partial T_{\theta r}}{\partial \theta} - \frac{T_{\theta\theta}}{r}\right)\mathbf{e}_r \\ &\quad + \left(\frac{1}{r}\frac{\partial}{\partial r}(rT_{r\theta}) + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{T_{\theta r}}{r})\right)\mathbf{e}_\theta. \end{aligned} \tag{63}$$

And, the gradient of $\mathbf{T}$ is the third order tensor

$$\begin{aligned} \operatorname{grad}\mathbf{T} &= \frac{\partial \mathbf{T}}{\partial r} \otimes \mathbf{e}_r + \frac{1}{r}\frac{\partial \mathbf{T}}{\partial \theta} \otimes \mathbf{e}_\theta \\ &= (\partial_r T_{rr})\,\mathbf{e}_r \otimes \mathbf{e}_r \otimes \mathbf{e}_r + (\partial_r T_{r\theta})\,\mathbf{e}_r \otimes \mathbf{e}_\theta \otimes \mathbf{e}_r \\ &\quad + (\partial_r T_{\theta r})\,\mathbf{e}_\theta \otimes \mathbf{e}_r \otimes \mathbf{e}_r + (\partial_r T_{\theta\theta})\,\mathbf{e}_\theta \otimes \mathbf{e}_\theta \otimes \mathbf{e}_r \\ &\quad + \left(\frac{1}{r}(\partial_\theta T_{rr}) - T_{r\theta} - T_{\theta r}\right)\mathbf{e}_r \otimes \mathbf{e}_r \otimes \mathbf{e}_\theta \\ &\quad + \left(\frac{1}{r}(\partial_\theta T_{\theta r}) + T_{rr} - T_{\theta\theta}\right)\mathbf{e}_\theta \otimes \mathbf{e}_r \otimes \mathbf{e}_\theta \\ &\quad + \left(\frac{1}{r}(\partial_\theta T_{r\theta}) + T_{rr} - T_{\theta\theta}\right)\mathbf{e}_r \otimes \mathbf{e}_\theta \otimes \mathbf{e}_\theta \\ &\quad + \left(\frac{1}{r}(\partial_\theta T_{\theta\theta}) + T_{r\theta} + T_{\theta r}\right)\mathbf{e}_\theta \otimes \mathbf{e}_\theta \otimes \mathbf{e}_\theta. \end{aligned}$$

2.4 Derivatives of scalar functions of tensors

We will also consider scalar functions of tensors and their derivatives with respect to a tensor. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be second-order tensors with Cartesian components in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ given by A_{ij}, B_{ij}, C_{ij}. Let $F = F(\mathbf{A})$ be a scalar function of $\mathbf{A}$. The derivative of the scalar function F with respect to the tensor $\mathbf{A}$ is a tensor with Cartesian components

$$\left(\frac{\partial F(\mathbf{A})}{\partial \mathbf{A}}\right)_{ij} = \frac{\partial F(\mathbf{A})}{\partial A_{ji}}. \tag{64}$$

That is

$$\frac{\partial F(\mathbf{A})}{\partial \mathbf{A}} = \frac{\partial F}{\partial A_{ji}}\mathbf{e}_i \otimes \mathbf{e}_j. \tag{65}$$

Now, let $\mathbf{A} = \mathbf{BC}$ and consider the derivative of F with respect to $\mathbf{B}$. In components, we have

$$\begin{aligned} \left(\frac{\partial F(\mathbf{A})}{\partial \mathbf{B}}\right)_{ij} &= \frac{\partial F(B_{kl}C_{lm})}{\partial B_{ji}} \\ &= \frac{\partial A_{km}}{\partial B_{ji}}\frac{\partial F(\mathbf{A})}{\partial A_{km}} = \frac{\partial B_{kl}C_{lm}}{\partial B_{ji}}\frac{\partial F(\mathbf{A})}{\partial A_{km}} \\ &= \delta_{jk}\delta_{il}C_{lm}\frac{\partial F(\mathbf{A})}{\partial A_{km}} = C_{im}\frac{\partial F(\mathbf{A})}{\partial A_{jm}} \\ &= C_{im}\left(\frac{\partial F(\mathbf{A})}{\partial \mathbf{A}}\right)_{mj}. \end{aligned} \tag{66}$$

So that, in general, we can write

$$\frac{\partial F(\mathbf{A})}{\partial \mathbf{B}} = \mathbf{C}\frac{\partial F(\mathbf{A})}{\partial \mathbf{A}}. \tag{67}$$

Other useful identities are *Jacobi's relations* for the first and second derivative of a non-vanishing determinant,

$$\frac{\partial}{\partial \mathbf{A}}(\det(\mathbf{A})) = \det(\mathbf{A})\mathbf{A}^{-1}, \tag{68}$$

$$\operatorname{tr}\left[\left(\frac{\partial}{\partial \mathbf{A}}\frac{\partial}{\partial \mathbf{A}}\det \mathbf{A}\right)\mathbf{B}\right] = \det(\mathbf{A})\left[\operatorname{tr}\left(\mathbf{A}^{-1}\mathbf{B}\right)\mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}\right], \tag{69}$$

where the contraction $\operatorname{tr}(\mathcal{L}\mathbf{A})$ of a second-order tensor $\mathbf{A}$ with a fourth-order tensor $\mathcal{L}$ is defined by $(\operatorname{tr}(\mathcal{L}\mathbf{A}))_{ij} = \mathcal{L}_{ijkl}A_{lk}$. In the last equality, the derivative of the inverse of a tensor by itself defines a fourth-order tensor such that

$$\operatorname{tr}\left[\left(\frac{\partial}{\partial \mathbf{A}}\mathbf{A}^{-1}\right)\mathbf{B}\right] = -\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}. \tag{70}$$

If $\mathbf{A} = \mathbf{A}(t)$, the derivative of a scalar function of $\mathbf{A}$ with respect to a parameter t can be obtained by the chain rule. That is,

$$\frac{\mathrm{d}}{\mathrm{d}t}F(\mathbf{A}) = \operatorname{tr}\left(\frac{\partial F(\mathbf{A})}{\partial \mathbf{A}}\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\right). \tag{71}$$

As an example, the first Jacobi relation (68) can be used to compute the derivative of the determinant of a tensor with respect to a parameter

$$\frac{\mathrm{d}}{\mathrm{d}t}(\det(\mathbf{A})) = \operatorname{tr}\left(\frac{\partial}{\partial \mathbf{A}}(\det(\mathbf{A}))\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\right) = \det(\mathbf{A})\operatorname{tr}\left(\mathbf{A}^{-1}\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\right). \tag{72}$$

2.5 The deformation gradient

The central geometric object of nonlinear elasticity that describes locally relative deformations is the *deformation gradient*, obtained as the spatial derivative of the mapping $\boldsymbol{\chi}$. Given a vector $\mathbf{x} = x_i(\mathbf{X})\mathbf{e}_i$, the deformation gradient tensor is $\mathbf{F} = \operatorname{Grad}\boldsymbol{\chi}$. In Cartesian coordinates, it reads

$$\mathbf{F} = \frac{\partial}{\partial X_j}(x_i\mathbf{e}_i) \otimes \mathbf{E}_j = \frac{\partial x_i}{\partial X_j}\mathbf{e}_i \otimes \mathbf{E}_j \equiv F_{ij}\mathbf{e}_i \otimes \mathbf{E}_j. \tag{73}$$

Note that the bases in which the gradient is taken are mixed. Geometrically, $\mathbf{F}$ is a linear map that transforms a vector $\mathbf{v}$ in the tangent space $T_p\mathcal{B}_0$ at a material point $p \in \mathcal{B}_0$ to a vector $\mathbf{F}\mathbf{v}$ in the tangent space $T_p\mathcal{B}$ at the same material point but in the current configuration (see Figure 4).

We can also express the deformation gradient in curvilinear coordinates. Let $\{q_1, q_2, q_3\}$ and $\{Q_1, Q_2, Q_3\}$ be the coordinates in the reference and current configuration, respectively. The deformation $\boldsymbol{\chi}$ in the bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ is given by $q_\alpha = q_\alpha(Q_1, Q_2, Q_3),\ \alpha = 1, 2, 3$. Then, following the definition of the gradient (58), we have

$$\begin{aligned}
\operatorname{Grad}\mathbf{x} &= H_\beta^{-1}\frac{\partial \mathbf{x}}{\partial Q_\beta} \otimes \mathbf{E}_\beta \\
&= H_\beta^{-1}\frac{\partial \mathbf{x}}{\partial q_\alpha}\frac{\partial q_\alpha}{\partial Q_\beta} \otimes \mathbf{E}_\beta && (74) \\
&= \frac{h_\alpha}{H_\beta}\frac{\partial q_\alpha}{\partial Q_\beta}\mathbf{e}_\alpha \otimes \mathbf{E}_\beta, && (75)
\end{aligned}$$

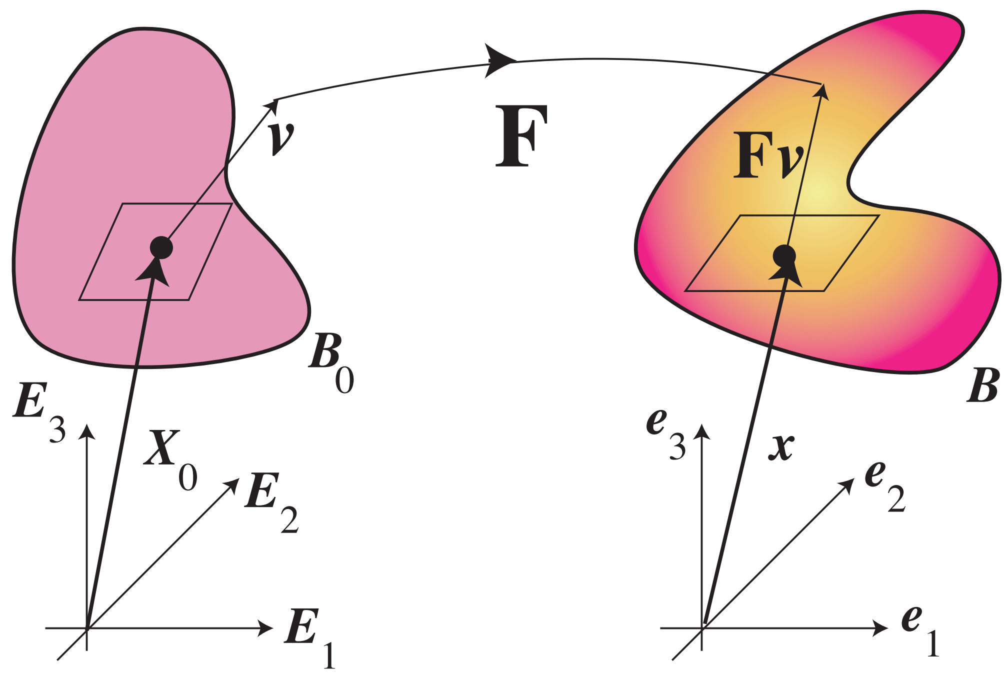


Figure 4: The deformation gradient maps vectors on the tangent space at a material point in the initial configuration to vectors in the tangent space in the current configuration at the same material point.

where we used $\frac{\partial \mathbf{x}}{\partial q_\beta} = h_\beta \mathbf{e}_\beta$ and the scale factors h_α, H_β given by (55). Then, we conclude that the matrix of coefficients of the deformation gradient $\mathbf{F} = F_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{E}_\beta$ is

$$[\mathbf{F}]_{\alpha\beta} = F_{\alpha\beta} = \frac{h_\alpha}{H_\beta}\frac{\partial q_\alpha}{\partial Q_\beta} \quad \text{(no summation on indices).} \tag{76}$$

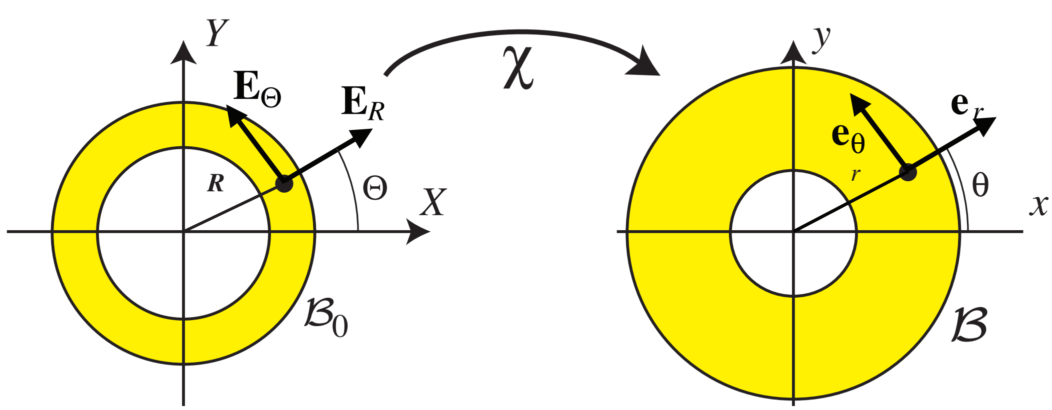


Figure 5: Deformation of a ring into another ring in the plane.

As an example, we consider the deformation of a ring in a plane to another ring shown in Figure 5. In the two sets of polar coordinates $\{q_1, q_2\} = \{r, \theta\}$ and $\{Q_1, Q_2\} = \{R, \Theta\}$, this deformation is given by

$$r = f(R), \quad \theta = \Theta. \tag{77}$$

For these coordinates, we have $h_r = H_R = 1$ and $h_\theta = r$, $H_\Theta = R$, and the deformation gradient is

$$[F] = \begin{bmatrix} \frac{h_r}{H_R}\frac{\partial r}{\partial R} & \frac{h_r}{H_\Theta}\frac{\partial r}{\partial \Theta} \\ \frac{h_\theta}{H_R}\frac{\partial \theta}{\partial R} & \frac{h_\theta}{H_\Theta}\frac{\partial \theta}{\partial \Theta} \end{bmatrix} = \begin{bmatrix} f'(R) & 0 \\ 0 & \frac{f(R)}{R} \end{bmatrix}, \tag{78}$$

that is,

$$\mathbf{F} = f'(R)\mathbf{e}_r \otimes \mathbf{E}_R + \frac{r}{R}\mathbf{e}_\theta \otimes \mathbf{E}_\Theta. \tag{79}$$

2.6 Volume, surface, and line elements

The deformation of a body may change the relative size of material elements. First, consider a set of material points in the reference configuration $\Omega_0 \subseteq \mathcal{B}_0$. This set evolves in time and is deformed to a new volume $\Omega \subseteq \mathcal{B}$ in the current configuration. The new volume is related to the reference volume by

$$\int_\Omega \mathrm{d}v = \int_{\Omega_0} J \mathrm{d}V_0, \tag{80}$$

where

$$J(\mathbf{X}, t) = \det \mathbf{F}(\mathbf{X}, t) \tag{81}$$

is the Jacobian of the transformation that represents the local change of volume, that is the image of an infinitesimal volume element $\mathrm{d}v$ at a material point p is

$$\mathrm{d}v = J\,\mathrm{d}V, \tag{82}$$

as shown in Figure 6.

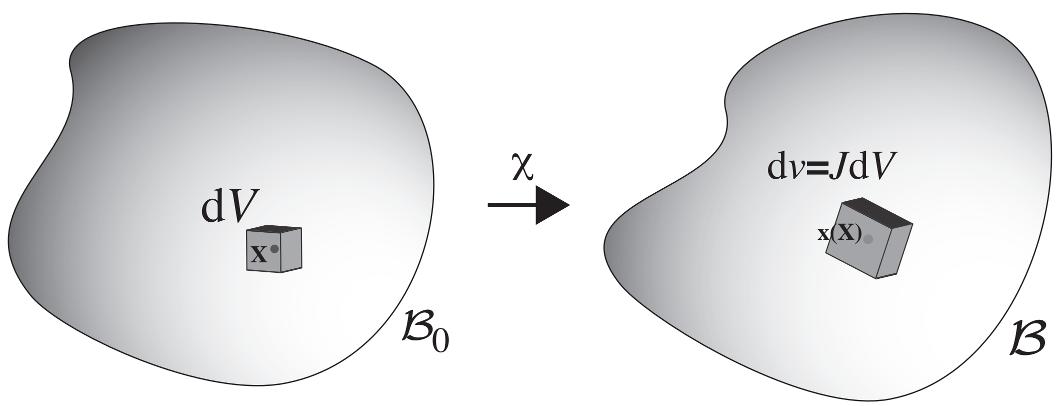


Figure 6: Transformation of volume: In a deformation, an infinitesimal volume element $\mathrm{d}V$ at a material point evolves by a factor J to a new volume $\mathrm{d}v = J\,\mathrm{d}V$.

Since volume elements are positive and cannot vanish in a deformation, we require that $J > 0$ in all deformations, which ensures the invertibility of the deformation gradient. That is, there exists a second-order tensor $\mathbf{F}^{-1}$ mapping vectors from $\mathcal{B}$ to $\mathcal{B}$, such that $\mathbf{F}^{-1}\mathbf{F} = \mathbf{1}$. Explicitly, this tensor is

$$\mathbf{F}^{-1} = \operatorname{grad} \mathbf{X}(\mathbf{x}, t). \tag{83}$$

A transformation that conserves locally every volume element, that is $J = 1$, is said to be *isochoric*.

Similarly, we can define an area element by considering a material area element normal to a given vector $\mathbf{N}$. Then, it is standard to show [118] that the surface integral transforms as

$$\int_{\partial\Omega} \mathbf{n}\,\mathrm{d}a = \int_{\partial\Omega_0} J\mathbf{F}^{-\mathsf{T}}\mathbf{N}\,\mathrm{d}A, \tag{84}$$

where $\mathbf{n}(\mathbf{x},t)$ and $\mathbf{N}(\mathbf{X},t)$ are outward unit normals, $\mathrm{d}A$ and $\mathrm{d}a$ are the area elements at a given point as shown in Figure 7. That is, an infinitesimal element of area defined in

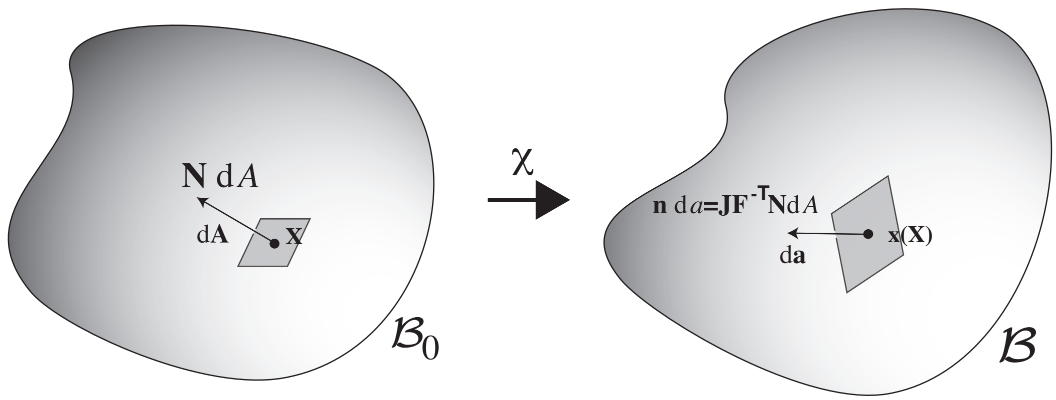


Figure 7: Transformation of area: In a deformation, an infinitesimal area is transformed according to Nanson's rule.

the reference configuration by a normal $\mathbf{N}$ and surface area $\mathrm{d}A$ is transformed into another element of area in the current configuration defined by a vector $\mathbf{n}$ with area $\mathrm{d}a$ and related to the reference one by *Nanson's formula*:

$$\mathbf{n}\,\mathrm{d}a = J\mathbf{F}^{-\mathsf{T}}\mathbf{N}\,\mathrm{d}A. \tag{85}$$

Finally, consider a local infinitesimal vector $\mathrm{d}\mathbf{X}$ tangent to a material line in $\mathcal{B}_0$ at a material point p, then its image is $\mathrm{d}\mathbf{x} = \mathbf{F}\mathrm{d}\mathbf{X}$ as shown in Figure 8. If $\mathbf{M}$ is the unit vector along $\mathrm{d}\mathbf{X}$ then

$$\mathrm{d}\mathbf{X} = \mathbf{M}\,\mathrm{d}S \quad \text{and} \quad \mathrm{d}\mathbf{x} = \mathbf{m}\,\mathrm{d}s, \tag{86}$$

where $\mathrm{d}S = |\mathrm{d}\mathbf{X}|$ and $\mathrm{d}s = |\mathrm{d}\mathbf{x}|$. This last identity implies that $\mathbf{m}\,\mathrm{d}s = \mathbf{F}\mathbf{M}\,\mathrm{d}S$. Now take the norm of each side:

$$|\mathrm{d}s|^2 = (\mathbf{F}\mathbf{M}\cdot\mathbf{F}\mathbf{M})|\mathrm{d}S|^2 = (\mathbf{F}^{\mathsf{T}}\mathbf{F}\mathbf{M})\cdot\mathbf{M}|\mathrm{d}S|^2. \tag{87}$$

Equivalently, we can write

$$\frac{\mathrm{d}s}{\mathrm{d}S} = \sqrt{(\mathbf{F}^{\mathsf{T}}\mathbf{F}\mathbf{M})\cdot\mathbf{M}}, \tag{88}$$

where $\mathrm{d}s/\mathrm{d}S$ is the change of length of a material line in the direction $\mathbf{M}$. This last relationship can be used to define the *stretch*, $\lambda = \lambda(\mathbf{M})$ of a material line in the direction $\mathbf{M}$ as

$$\lambda(\mathbf{M}) = \sqrt{(\mathbf{F}^{\mathsf{T}}\mathbf{F}\mathbf{M})\cdot\mathbf{M}}. \tag{89}$$

Since we are interested in characterizing elastic materials, we need to characterize deformations which change the relative length of line elements. Therefore, stretches provides a

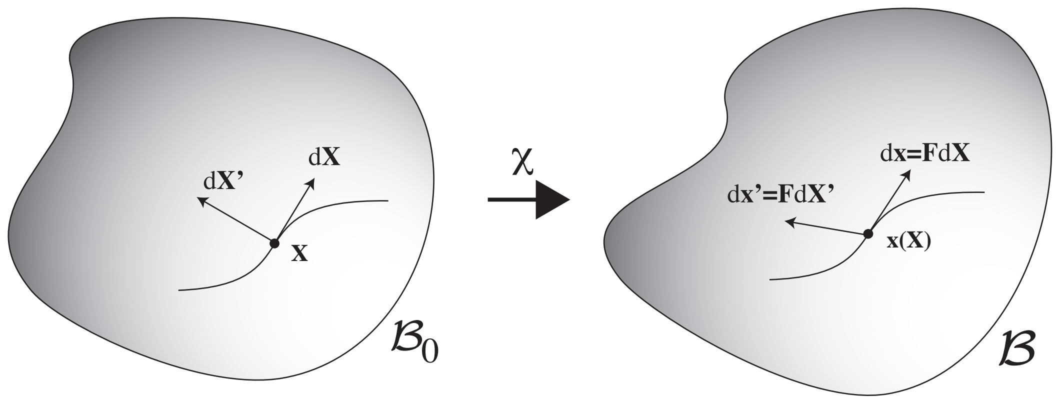


Figure 8: Transformation of material lines: In a deformation, an infinitesimal line element is mapped by the deformation gradient to a new line element.

natural measure of *strain* in a material. A material is said to be *unstrained* in the direction $\mathbf{M}$ if and only if $\lambda(\mathbf{M}) = 1$.

In (87), we see the appearance of an important tensor in the description of strain for a three-dimensional body, namely the *right Cauchy-Green tensor*

$$\mathbf{C} = \mathbf{F}^{\mathsf{T}}\mathbf{F}. \tag{90}$$

A material is *unstrained* at a given point if it is unstrained in all directions. That is $\lambda(\mathbf{M}) = 1\ \forall \mathbf{M}$. In terms of the right Cauchy-Green tensor, this implies $\mathbf{C} = \mathbf{1}$.

Geometrically, the tensor $\mathbf{C}$ can be interpreted as a metric on $\mathcal{B}$ as it provides a way to measure distances and angles on the new body (see Section **??**).

2.7 Polar decomposition theorem

The action of the deformation gradient $\mathbf{F}$ on a vector $\mathbf{M}$ can be decomposed into a rotation about a direction $\mathbf{m}$, followed by a stretch of size $\lambda(\mathbf{M})$. This decomposition into a stretch and a rotation can be applied directly to the deformation gradient through the polar decomposition theorem stating that: *For a second-order tensor* $\mathbf{F}$ *such that* $\det \mathbf{F} > 0$, *there exist unique positive definite symmetric tensors* $\mathbf{U}$, $\mathbf{V}$ *and a unique proper orthogonal tensor* $\mathbf{R}$ *such that,*

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}.$$

The positive symmetric tensors $\mathbf{U}$ and $\mathbf{V}$ are called the *left* and *right stretch tensors*, respectively. Their squares can easily be obtained from $\mathbf{F}$ as follows

$$\begin{aligned} \mathbf{F}^{\mathsf{T}}\mathbf{F} &= \mathbf{U}^2 \equiv \mathbf{C}, && \text{the right Cauchy-Green tensor,} \\ \mathbf{F}\mathbf{F}^{\mathsf{T}} &= \mathbf{V}^2 \equiv \mathbf{B}, && \text{the left Cauchy-Green tensor.} \end{aligned}$$

Since $\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^{\mathsf{T}}$, $\mathbf{U}$ and $\mathbf{V}$ have the same eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$. The *principal stretches*, can be obtained conveniently as the square roots of the eigenvalues of either $\mathbf{C}$ or $\mathbf{B}$. Note that since $\mathbf{U}$ and $\mathbf{V}$ are positive symmetric, the principal stretches are positive and real and the corresponding eigenvectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathbf{U}$ and $\mathbf{V}$ form two bases.

Therefore, the stretch tensors can be written as

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}_i \otimes \mathbf{u}_i, \tag{91}$$

$$\mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i. \tag{92}$$

Since $J = \det \mathbf{F} = \det(\mathbf{RU}) = (\det \mathbf{R})(\det \mathbf{U}) = \det \mathbf{U} = \det \mathbf{V}$, we have $J = \lambda_1\lambda_2\lambda_3$ and we can re-write $\mathbf{F}$ as

$$\mathbf{F} = \sum_{i=1}^{3} \lambda_i \mathbf{v}_i \otimes \mathbf{u}_i. \tag{93}$$

2.8 Velocity, acceleration, and velocity gradient

The motion associated with a deformation $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, $\mathbf{X} \in \mathcal{B}_0$, is associated with change in time t. Since $\mathbf{X}$ is the position of a material point, the *velocity* and *acceleration* of this material point are, respectively,

$$\mathbf{v}(\mathbf{x}, t) = \frac{\partial}{\partial t}\boldsymbol{\chi}(\mathbf{X}, t) \equiv \dot{\boldsymbol{\chi}}(\mathbf{X}, t), \tag{94}$$

$$\mathbf{a}(\mathbf{x}, t) = \frac{\partial^2}{\partial t^2}\boldsymbol{\chi}(\mathbf{X}, t) \equiv \ddot{\boldsymbol{\chi}}(\mathbf{X}, t). \tag{95}$$

In general, we define the *material time derivative* $\mathrm{d}/\mathrm{d}t$ as a total time derivative with respect to a fixed material coordinate $\mathbf{X}$. For a scalar field $\phi = \phi(\mathbf{x}, t)$, the material derivative is

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi \equiv \left.\frac{\mathrm{d}\phi}{\mathrm{d}t}\right|_{\mathbf{X}} \equiv \dot{\phi} \equiv \frac{\partial \phi}{\partial t} + (\mathrm{grad}\phi) \cdot \mathbf{v}, \tag{96}$$

and we define the derivative of a vector field $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ similarly as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \frac{\partial \mathbf{u}}{\partial t} + (\mathrm{grad}\, \mathbf{u})\, \mathbf{v}. \tag{97}$$

Another important kinematic quantity is the *velocity gradient tensor*, defined as

$$\mathbf{L} = \mathrm{grad}\, \mathbf{v}, \qquad L_{ij} = \frac{\partial v_i}{\partial x_j}, \qquad \mathbf{L} = L_{ij}\mathbf{e}_i \otimes \mathbf{e}_j. \tag{98}$$

Since, in general the chain rule gives $\mathrm{Grad}\, \mathbf{u} = (\mathrm{grad}\, \mathbf{u})\mathbf{F}$, we have

$$\mathrm{Grad}\, \mathbf{v} = (\mathrm{grad}\, \mathbf{v})\mathbf{F} = \mathbf{LF}, \tag{99}$$

but also,

$$\mathrm{Grad}\, \mathbf{v} = \mathrm{Grad}\, \dot{\mathbf{x}} = \frac{\partial}{\partial t}\mathrm{Grad}\, \mathbf{x} = \frac{\partial \mathbf{F}}{\partial t} = \dot{\mathbf{F}}, \tag{100}$$

so that the evolution of the deformation gradient can be expressed in terms of the velocity gradient tensor as

$$\dot{\mathbf{F}} = \mathbf{LF}. \tag{101}$$

Taking the determinant of each side of this last equality and applying Jacobi's formula (72) for the derivative of the determinant of a nonsingular matrix

$$\frac{\partial}{\partial t} \det \mathbf{F} = (\det \mathbf{F}) \operatorname{tr}(\mathbf{F}^{-1}\dot{\mathbf{F}}) = (\det \mathbf{F}) \operatorname{tr}(\mathbf{L}) \tag{102}$$

leads to an equation for the evolution of the determinant:

$$\dot{J} = J \operatorname{tr}(\mathbf{L}) = J \operatorname{div} \mathbf{v}. \tag{103}$$

In particular, since $J \neq 0$, we note the well-known relationship between conservation of volume during motion and the vanishing of the divergence of the velocity:

$$\operatorname{div} \mathbf{v} = 0 \quad \iff \quad \dot{J} = 0. \tag{104}$$

2.9 Examples of deformation

2.9.1 Homogeneous deformation

$$\mathbf{x} = \mathbf{F}\mathbf{X} + \mathbf{x}, \qquad \mathbf{F} \text{ constant} \tag{105}$$

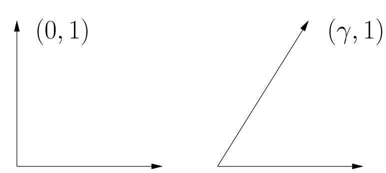
- *Simple elongation*

$$\mathbf{F} = \mathbf{U} = \lambda_1 \mathbf{U}^{(1)} \otimes \mathbf{U}^{(1)} + \lambda_2 \left(\mathbf{U}^{(2)} \otimes \mathbf{U}^{(2)} + \mathbf{U}^{(3)} \otimes \mathbf{U}^{(3)} \right) \tag{106}$$

- *Dilation*

$$\mathbf{F} = \lambda \, \mathbb{1} \tag{107}$$

- *Simple shear*



$$x_1 = X_1 + \gamma X_2, \qquad x_2 = X_2, \qquad x_3 = X_3, \tag{108}$$

which imply (homework),

$$\Longrightarrow \quad \mathbf{F} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{U}^2 = \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{109}$$

2.9.2 Inflation of a spherical shell

We consider the symmetric deformation of a spherical shell with radii $R = A$ and $R = B$ in the initial configuration into a spherical shell of radii $r = a$ and $r = b$ in the current configuration. We consider the possibility where both A and a vanish (a sphere under load); the case where $A = 0$ but $a > 0$ (the *cavitation* of a sphere); as well as the case $A > 0$ and $a = 0$ (the *anti-cavitation* of a spherical shell). Under the deformation, the shell expands and any point located at (R, Θ, Φ) in the reference configuration is moved to the point (r, Θ, Φ) where $r = r(R)$ as shown in Figure 9. Explicitly, the deformation, $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$, reads

$$r = r(R), \quad \theta = \Theta, \quad \phi = \Phi, \tag{110}$$

so that the position vectors are, respectively,

$$\mathbf{X} = R\mathbf{E}_R, \qquad \mathbf{x} = r(R)\mathbf{e}_r = \frac{r(R)}{R}\mathbf{X}. \tag{111}$$

Due to the symmetry of the deformation, we can identify the basis vectors so that $\mathbf{E}_R = \mathbf{e}_r, \mathbf{E}_\Theta = \mathbf{e}_\theta, \mathbf{E}_\Phi = \mathbf{e}_\phi$.

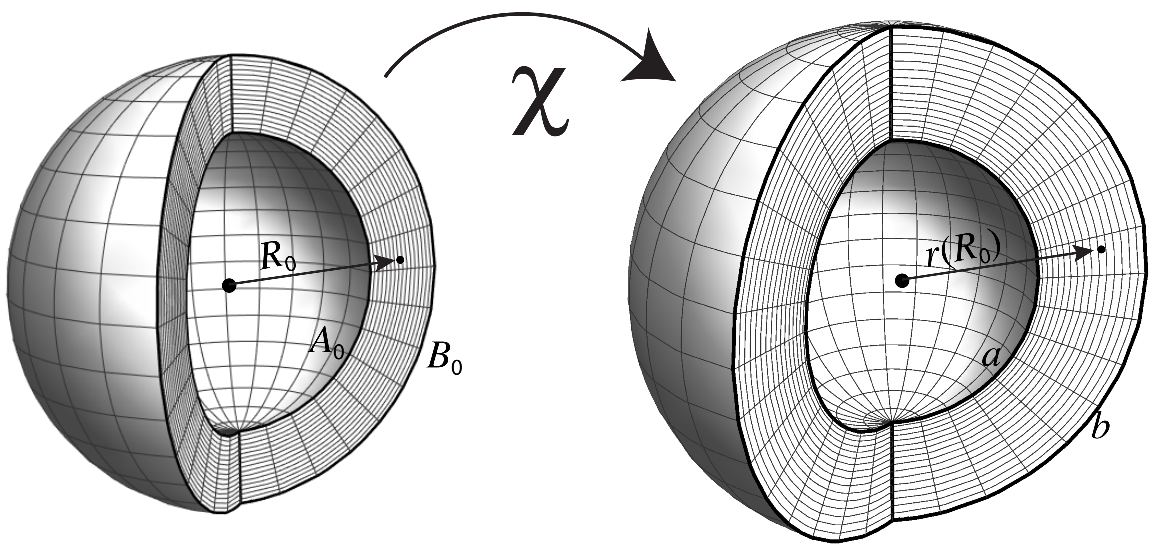


Figure 9: Radial deformation of a shell with inner and outer radii A and B to a shell with radii a and b.

To compute the deformation gradient, we use the method described in Section 2.5. We have two sets the two sets of spherical coordinates $\{q_\alpha\} = \{r, \theta, \phi\}$ and $\{Q_\alpha\} = \{R, \Theta, \Phi\}$. For these coordinates, the scale factors, defined by (55), are

$$h_r = 1, \quad H_R = 1, \tag{112}$$
$$h_\theta = r, \quad H_\Theta = R, \tag{113}$$
$$h_\phi = r, \quad H_\Phi = R. \tag{114}$$

Following Equation (76), the deformation gradient is

$$\mathbf{F} \;=\; r'\mathbf{e}_r \otimes \mathbf{e}_r + \frac{r}{R}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \frac{r}{R}\mathbf{e}_\phi \otimes \mathbf{e}_\phi,$$

which we write

$$\mathbf{F} = \operatorname{diag}(r', r/R, r/R), \tag{115}$$

where the primes denote derivatives with respect to R.

Note if we only consider *isochoric* deformation then $\det \mathbf{F} = 1$, which implies

$$r'\left(\frac{r}{R}\right)^2 = 1 \quad \Longrightarrow \quad r'r^2 = R^2 \quad \Longleftrightarrow \quad \frac{1}{3}\frac{\mathrm{d}(r^3)}{\mathrm{d}R} = R^2 \quad \Longrightarrow \quad r^3 = R^3 + C. \tag{116}$$

Since $r(a) = A$, $r(b) = B$,

$$C = b^3 - B^3 = a^3 - A^3 \quad \Longrightarrow \quad a^3 = b^3 - B^3 + A^3 \quad \Longrightarrow \quad r = \sqrt{a^3 - A^3 + R^3} \tag{117}$$

This is a one-parameter family of solutions.

3 Conservation Laws, Stress, and Dynamics

■ **Overview** We use basic physical principles to derive local equations for the evolution of mass and stress in space. The requirement for localising the balance of linear momentum naturally leads to the definition of the Cauchy stress tensor.

We have obtained a complete description of the deformation of a body. Starting with a mapping $\boldsymbol{\chi}$, we defined the deformation gradient $\mathbf{F}$. This tensor contains all information on relative deformation of a body, such as local changes of volume, area, and stretch. It was used to define secondary quantities, such as the left and right Cauchy-Green tensor, $\mathbf{B}$ and $\mathbf{C}$, that contain information on the strain developed during a deformation. Now that we have a complete kinematic description of the of the body, we can define physical fields at each point on the body and used fundamental laws of physics to find local equations between these fields.

The governing equations of continuum mechanics are obtained by considering the local balance of physical quantities: mass, linear momentum, angular momentum, and energy. The traditional approach to derive local laws consists in stating a balance law on an arbitrary subset of the body and, under suitable conditions, obtaining local relationships between physical quantities expressed as differential equations. We illustrate this process first on the balance of mass.

3.1 Balance of mass

To describe the properties and response of a material, we attach physical quantities at each point of the body B. First, we define a scalar field $\rho = \rho(\mathbf{x}, t)$, the *volume density* (mass per unit current volume) at each point of the body in the current configuration and assume that the mass of any subset of the body $\Omega \subseteq \mathcal{B}$ is conserved in time, that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \rho\,(\mathbf{x}, t)\,\mathrm{d}v = 0. \tag{118}$$

The problem here is that the position of a material subset Ω evolves with time, and the time derivative cannot be directly applied to the integrand. Therefore, we first map the integral to the reference configuration

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_0} \rho\,(\mathbf{x}(\mathbf{X}, t), t)\,J\mathrm{d}V = 0, \tag{119}$$

where we have used the transformation of a volume element (82).

Second, since the domain Ω_0 is fixed, we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_0} J\rho\,(\mathbf{x}, t)\,\mathrm{d}V = \int_{\Omega_0}\frac{\mathrm{d}}{\mathrm{d}t}\,(J\rho)\,\mathrm{d}V = 0. \tag{120}$$

Third, since we wish to obtain a balance law in the current configuration, we map the integral back, that is

$$\int_{\Omega_0}\frac{\mathrm{d}}{\mathrm{d}t}\,(J\rho)\,\mathrm{d}V = \int_{\Omega}\frac{\mathrm{d}}{\mathrm{d}t}\,(J\rho)\,J^{-1}\mathrm{d}v = \int_{\Omega}(\dot{\rho} + \rho\,\mathrm{div}\,\mathbf{v})\,\mathrm{d}v = 0, \tag{121}$$

where we have used (103).

Fourth, assuming that the integrand is continuous, the vanishing of an integral on an arbitrary domain implies that it vanishes pointwise, which leads to the usual *continuity equation* for the evolution of density in the current configuration [60]

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0. \tag{122}$$

The procedure consisting in mapping an integral relationship between configurations and localizing this relationship to obtain a local differential equation is a two-step process called the *Maxwell transport* and *localization procedure*, respectively.

In the first step, integrals in the current configuration are transformed into integrals in the initial configuration that are expressed on a fixed domain. In this configuration, the balance law can be written as a single integral over a domain. Once this expression has been obtained, the integral can be mapped back to the current configuration.

In the second step, a local differential equation is obtained from the integral by assuming that it holds on an arbitrary subset and that the integrand is continuous.

Note that the localization procedure can be also applied directly to the first integral appearing in (121). That is

$$\frac{\mathrm{d}}{\mathrm{d}t} (J\rho) = 0. \tag{123}$$

If we define the *reference density* $\rho_0 (\mathbf{X}, t) = J (\mathbf{X}, t) \rho (\mathbf{x}(\mathbf{X}, t), t)$, then the *mass conservation* in the reference configuration is simply

$$\frac{\partial}{\partial t} \rho_0 = 0. \tag{124}$$

3.1.1 Transport formulas

We will be making systematic use of the Maxwell transport to obtain balance laws. It is therefore important to obtain general transport relationships for for any scalar ϕ or vector field $\mathbf{u}$ associated with the moving body in the current configuration. Indeed, following the same steps as in (119-121), we obtain [120] the useful transport formulas:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \phi \, \mathrm{d}v = \int_\Omega (\dot{\phi} + (\operatorname{div} \mathbf{v})\phi) \, \mathrm{d}v, \tag{125}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \mathbf{u} \, \mathrm{d}v = \int_\Omega (\dot{\mathbf{u}} + (\operatorname{div} \mathbf{v})\mathbf{u}) \, \mathrm{d}v, \tag{126}$$

where $\Omega \subseteq \mathcal{B}$ is an arbitrary subset.

3.2 Balance of linear momentum

The balance of linear momentum expresses the fundamental relationship between the rate of change of linear momentum of a body as a result of the force applied to the body. When applied to a rigid body, it simply leads to the well-know Newton's second law. However, in the case of a deformable body, the body also experiences internal forces that need to be taken into account.

In the current configuration, the total linear momentum on any part of the body $\Omega \subseteq \mathcal{B}$ is simply

$$\int_\Omega \rho(\mathbf{x}, t) \mathbf{v}(\mathbf{x}, t) \mathrm{d}v, \tag{127}$$

where $\rho(\mathbf{x}, t)$ is the density and $\mathbf{v}(\mathbf{x}, t)$ the velocity of a point at $(\mathbf{x}, t)$.

The total force acting on a point $\mathbf{x} \in \Omega$ includes a *body-force* density $\mathbf{b}$, representing the contributions of external forces and a *contact-force* density $\mathbf{t}_n$, representing the force per unit area resulting in contact. Therefore, the total force acting on Ω is

$$\int_\Omega \rho(\mathbf{x},t)\mathbf{b}(\mathbf{x},t)\mathrm{d}v + \int_{\partial\Omega} \mathbf{t}_n \mathrm{d}a. \tag{128}$$

Euler's first law of motion then states that the rate of change of the linear momentum on any part of the body $\Omega \subseteq \mathcal{B}$ is equal to the sum of the forces acting on Ω [146]. Therefore, in our context, this law reads

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)\mathrm{d}v}_{\text{rate of change of linear momentum}} = \underbrace{\int_\Omega \rho(\mathbf{x},t)\mathbf{b}(\mathbf{x},t)\mathrm{d}v + \int_{\partial\Omega} \mathbf{t}_n \mathrm{d}a}_{\text{sum of body and contact forces}}. \tag{129}$$

We apply the transport formula (126) by mapping the rate of linear momentum to the reference configuration

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho\mathbf{v}\,\mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_0} \rho\mathbf{v}J\,\mathrm{d}V = \int_{\Omega_0} \frac{\mathrm{d}}{\mathrm{d}t}(\rho\mathbf{v}J)\,\mathrm{d}V, \tag{130}$$

$$= \int_{\Omega_0} (\rho\dot{\mathbf{v}} + \underbrace{\dot{\rho}\mathbf{v} + \rho\mathbf{v}\,\mathrm{div}\,\mathbf{v}}_{=0,\text{ per continuity}})\,J\mathrm{d}V, \tag{131}$$

$$= \int_\Omega \rho\dot{\mathbf{v}}\,\mathrm{d}v, \tag{132}$$

where we have used the identity (103).

In order to apply the localization procedure, we need to express all the quantities as a single integral. However, the last integral in (129) is a surface integral. Therefore, it needs to be expressed as a volume integral. The standard way to transform a surface integral as a volume integral is to use the divergence theorem. However, the integrand does not have the required form for a direct application of the divergence theorem. This difficulty prompts us to re-express the contact-force density $\mathbf{t}_\mathrm{n}$ in a tensorial form.

We first use the *Cauchy's stress principle* stating that the contact-force density depends continuously on the unit normal $\mathbf{n}$. Then, Cauchy's tetrahedral argument [118] can be used to show that the contact-force density depends linearly on the unit normal, so that

$$\mathbf{t}_n = \mathbf{T}\mathbf{n}, \tag{133}$$

where $\mathbf{T}$ is a second-order tensor independent of $\mathbf{n}$. This last identity has the correct form for the application of the divergence theorem (46):

$$\int_{\partial\Omega} \mathbf{t}_\mathrm{n}\,\mathrm{d}a = \int_{\partial\Omega} \mathbf{T}\mathbf{n}\,\mathrm{d}a = \int_\Omega \mathrm{div}\,(\mathbf{T}^\mathsf{T})\,\mathrm{d}v \tag{134}$$

The tensor $\mathbf{T}$ is the *Cauchy stress tensor*, a central quantity describing forces per unit area in a material. Using this last equality, Euler's law (129) simplifies to

$$\int_\Omega \rho\dot{\mathbf{v}}\mathrm{d}v = \int_\Omega \rho\mathbf{b}\mathrm{d}v + \int_\Omega \mathrm{div}\,(\mathbf{T}^\mathsf{T})\,\mathrm{d}v, \tag{135}$$

and the localization procedure leads to the first *Cauchy equation*:

$$\mathrm{div}\,(\mathbf{T}^\mathsf{T}) + \rho\mathbf{b} = \rho\dot{\mathbf{v}}. \tag{136}$$

3.3 Balance of angular momentum

In a continuum both forces and torques have to be balanced. The corresponding balance for torque is Euler's second law of motion stating that the rate of change of angular momentum of an arbitrary material subset $\Omega \subseteq \mathcal{B}$ with respect to a given point is equal to the sum of all torques acting on Ω with respect to the same point.

The total angular momentum of Ω is

$$\int_\Omega \rho \mathbf{x} \times \mathbf{v} \mathrm{d}v, \tag{137}$$

where, without loss of generality, we choose to express the angular momentum with respect to the origin.

If we assume here that the material under consideration is *non-polar*, that is the body is not subject to extra body or contact torques and cannot support couple stresses, then the total torque due to body and traction forces acting on Ω, with respect to the origin, is

$$\int_\Omega \rho \mathbf{x} \times \mathbf{b}\, \mathrm{d}v + \int_{\partial\Omega} \mathbf{x} \times \mathbf{t}_n\, \mathrm{d}a. \tag{138}$$

Then, Euler's second law can be written as

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho \mathbf{x} \times \mathbf{v} \mathrm{d}v}_{\text{rate of change of angular momentum}} = \underbrace{\int_\Omega \rho \mathbf{x} \times \mathbf{b}\, \mathrm{d}v + \int_{\partial\Omega} \mathbf{x} \times \mathbf{t}_n\, \mathrm{d}a.}_{\text{torques due to body and traction forces}} \tag{139}$$

The transport procedure and the continuity equation can be used to simplify this expression to

$$\int_\Omega \rho \mathbf{x} \times (\dot{\mathbf{v}} - \mathbf{b})\, \mathrm{d}v = \int_{\partial\Omega} \mathbf{x} \times \mathbf{T}\mathbf{n}\, \mathrm{d}a, \tag{140}$$

where we have expressed the contact force in terms of the Cauchy stress tensor, $\mathbf{t}_n = \mathbf{T}\mathbf{n}$. Cauchy's first equation (136) can be used to transform the right hand side of this expression to obtain

$$\int_\Omega \mathbf{x} \times \operatorname{div}(\mathbf{T}^\mathsf{T})\, \mathrm{d}v = \int_{\partial\Omega} \mathbf{x} \times \mathbf{T}\mathbf{n}\, \mathrm{d}a. \tag{141}$$

By application of the divergence theorem and the localization procedure, the last integral implies that the Cauchy stress tensor is symmetric, that is

$$\mathbf{T}^\mathsf{T} = \mathbf{T}. \tag{142}$$

We can now simplify (136) to obtain the standard form of Cauchy's equation, also known as *the equation of motion* for a continuum:

$$\operatorname{div} \mathbf{T} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}, \tag{143}$$

3.4 Many stress tensors

The Cauchy stress tensor is the natural measure for contact forces measured in the current configuration per unit area in the current configuration. The contact forces acting on the boundary $\partial\Omega$ of a region Ω can be extracted from $\mathbf{T}$ by considering different directions with respect to the boundary. Let $\mathbf{n}$ be the normal vector to $\partial\Omega$ at a point p in the body. Then, recalling that $\mathbf{t}_n = T\mathbf{n}$, the *normal stress*, that is, the force per area normal to $\partial\Omega$, applied at p is

$$\mathbf{n} \cdot \mathbf{t}_n = \mathbf{n} \cdot (\mathbf{T}\mathbf{n}). \tag{144}$$

Considering a vector $\mathbf{m}$ tangent to $\partial\Omega$ at p (that is $\mathbf{m} \cdot \mathbf{n} = 0$), the product

$$\mathbf{m} \cdot \mathbf{t}_n = \mathbf{m} \cdot (\mathbf{T}\mathbf{n}). \tag{145}$$

is a *shear stress* acting on Ω at p.

While the Cauchy stress is a natural measure for forces acting on a continuum it is not always a convenient quantity for computation since the area in the current configuration changes during the deformation. Therefore, it is often useful to measure contact forces with respect to areas measured initially in the reference configuration. To do so, we apply Nanson's formula $\mathbf{n}\,\mathrm{d}a = J\mathbf{F}^{-\mathsf{T}}\mathbf{N}\,\mathrm{d}A$ to the traction vector to obtain the contact-force on a material area element:

$$\mathbf{t}_n\,\mathrm{d}a = \mathbf{T}\mathbf{n}\,\mathrm{d}a = (J\mathbf{T}\mathbf{F}^{-\mathsf{T}})\mathbf{N}\,\mathrm{d}A = \mathbf{S}^{\mathsf{T}}\,\mathbf{N}\,\mathrm{d}A, \tag{146}$$

where

$$\mathbf{S} = J\mathbf{F}^{-1}\mathbf{T} \tag{147}$$

is the *nominal stress tensor*. Its transpose, $\mathbf{S}^{\mathsf{T}}$, is the *first Piola-Kirchhoff stress tensor*. It is also called the *engineering stress tensor*, as it is a convenient quantity for experimental measurements.

Since $\mathbf{T}$ is symmetric, we have

$$\mathbf{S}^{\mathsf{T}}\mathbf{F}^{\mathsf{T}} = \mathbf{F}\mathbf{S}. \tag{148}$$

3.5 Balance of energy for elastic materials

The equations for the stress and mass density derived so far are valid for a large class of continuum bodies independently of their specific material characteristics, including solids and fluids. To close the system of equations, constitutive relationships between stress, deformation gradient, rate of deformation and density must be imposed to characterize the particular body under consideration. The balance of energy provides restriction on the form of these constitutive relationships. Here, we turn our attention to elastic materials.

The general principle for the balance energy states that for any part of a body $\Omega \subseteq \mathcal{B}$, the rate of change of the total mechanical energy $\mathcal{E}$ is balanced by the power of the forces $\mathcal{P}$. If we ignore heat dissipation, the total energy for an elastic material is the sum of the kinetic energy and an internal elastic energy. That is, we have

$$\mathcal{E} = \underbrace{\frac{1}{2}\int_\Omega \rho\mathbf{v} \cdot \mathbf{v}\,\mathrm{d}v}_{\text{kinetic energy}} + \underbrace{\int_\Omega J^{-1}W\,\mathrm{d}v}_{\text{internal energy}}, \tag{149}$$

where W is the internal elastic energy density per unit reference volume.

The power of the forces acting on Ω is given by

$$\mathcal{P} = \int_\Omega \rho\,\mathbf{b} \cdot \mathbf{v}\,\mathrm{d}v + \int_{\partial\Omega} \mathbf{t}_n \cdot \mathbf{v}\,\mathrm{d}a. \tag{150}$$

The balance of energy is then

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = \mathcal{P}. \tag{151}$$

The transport and localization procedure applied to the energy principle together with the equations of continuity and motion, lead to a local form of energy balance [118] for the *stress power*

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \mathrm{tr}(\mathbf{S}\dot{\mathbf{F}}). \tag{152}$$

From a thermodynamic point of view, the combination of $\mathbf{S}$ and $\mathbf{F}$ in the stress power $\mathrm{tr}(\mathbf{S}\dot{\mathbf{F}})$, identifies these two tensors as being *work conjugate*, that is they form a conjugate pair of stress and deformation tensors. We will use this important identity later on when we further restrict the dependence of the internal energy.

4 Constitutive equations

■ **Overview** We close the system of Cauchy equations for stress, density, and velocity by introducing a relationship between stresses and strains, the constitutive equations. Depending on the choice of constitutive equations, the continuum equations can describe a fluid, a solid, or a gas.

So far, we have obtained through conservation of mass and balance of momenta, the following three equations

$$\dot{\rho} + \rho \text{div } \mathbf{v} = 0, \qquad \text{mass} \tag{153}$$

$$\text{div } \mathbf{T} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}, \qquad \text{linear momentum} \tag{154}$$

$$\mathbf{T}^T = \mathbf{T}, \qquad \text{angular momentum} \tag{155}$$

There are 10 unknowns: 1 in ρ, 3 in vector $\mathbf{v}$ and 6 in the symmetric tensor $\mathbf{T}$. But there are only 4 equations. We need 6 extra relationships to close this system. These will be given by the constitutive equations.

4.1 3 types of assumptions

1) *Possible deformations.*
   e.g. Only rigid motions are allowed ($\mathbf{F} = \mathbf{R}$, 3 parameters). $\implies$ rigid body mechanics.
   e.g. Only isochoric motion $\implies$ Incompressible material.

2) *Constraining the stress tensor*
   e.g. $\mathbf{T} = \mathcal{T}(\mathbf{F})$
   e.g. $\mathbf{T} = -p\mathbb{1}$

3) *Relate stress to motion*
   e.g. pressure function of density, ρ (for a gas).

4.1.1 Particular examples

1) *Ideal fluids*

   (a) $\det \mathbf{F} = 1$ (Isochoric)

   (b) $\rho = \text{const}$

   (c) $\mathbf{T} = -p\mathbb{1}$

   Note: the pressure is *not* determined by the motion (ball under uniform pressure). (Lagrange multiplier for the pressure.)

2) *Elastic fluids*

   (a) $\mathbf{T} = -p\mathbb{1}$

   (b) $p = p(\rho)$

   Here $\ddot{\rho} = P'(\rho_0)\Delta\rho$ and $\sqrt{p'}$ is the sound speed.
   N.B.: both fluids are inviscid (do not exert shearing forces!)

   A particular case of an elastic fluids is an *ideal gas*: $p = \lambda\rho^\gamma$, for $\lambda > 0$, $\gamma > 1$.

3) *Newtonian fluids.* Shear stress through friction.
Take $\mathbf{L} = \text{grad } \mathbf{v}$ which gives relative motion of particles, velocity gradient.

(a) $\det \mathbf{F} = 1$, incompressible

(b) $\mathbf{T} = -p\mathbb{1} + \mathcal{C}[\mathbf{L}]$ where $\mathcal{C}$ is a linear function of $\mathbf{L}$.

Note $\mathcal{C}[0] = 0 \implies \mathbf{T} = -p\mathbb{1}$, A Newtonian fluid at rest is ideal
Note $\mathcal{C}[\mathbf{L}]$ has 40 independent constants (once we have removed arbitrariness of $p\mathbb{1}$. However *objectivity* (independence from observer) implies

$$\mathcal{C}[\mathbf{L}] = 2\mu\mathbf{D}, \qquad \mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T), \tag{156}$$

which has a single constant, viscosity μ. This implies

$$\rho\dot{\mathbf{v}} = \text{div } \mathbf{T} + \rho\mathbf{b} \tag{157}$$

$$\text{div } \mathbf{v} = 0 \tag{158}$$

$$\mathbf{T} = -p\mathbb{1} + 2\mu\mathbf{D} \tag{159}$$

After some algebra,

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho\mathbf{v} \cdot \text{grad } \mathbf{v} = \mu\Delta - \text{grad } p + \rho\mathbf{b}, \tag{160}$$

$$\text{div } \mathbf{v} = 0, \tag{161}$$

which are the Navier–Stokes equations. (N.B. $\nu = \mu/\rho$ is the kinematic viscosity.)

Stokes flow: 1) steady, 2) neglect acceleration.

$$\Delta\mathbf{v} = \text{grad } p - \mathbf{b} \tag{162}$$

$$\text{div } \mathbf{v} = 0. \tag{163}$$

N.B. for more general fluids, $\mathbf{T} = -p\mathbb{1} + \mathcal{N}(\mathbf{L})$.

4.2 Elastic materials

For elastic materials, we have the simple relationship

$$\mathbf{T} = \mathcal{Z}(\mathbf{F}) \tag{164}$$

This implies that the stress in $\mathcal{B}$ at $\mathbf{x}$ depends on $\mathbf{F}$ and not on the history of the deformation (path-independent). Also, by the definition of the reference configuration (assuming that it is stress free), we have

$$\mathcal{Z}(\mathbb{1}) = 0. \tag{165}$$

This relationship defines *a Cauchy, elastic material.*

4.3 Constitutive equations for hyperelastic materials

We further assume that the material is *hyperelastic*. That is, the internal energy density W is a function of $\mathbf{F}$ alone. Explicitly, we posit that

$$W(\mathbf{X}, t) = W(\mathbf{F}(\mathbf{X}, t), \mathbf{X}), \tag{166}$$

in which case, W is referred to as the *strain-energy function* of the system. Using (71), the time derivative of W is

$$\frac{\mathrm{d}}{\mathrm{d}t} W(\mathbf{F}) = \mathrm{tr}\left(\frac{\partial W}{\partial \mathbf{F}} \dot{\mathbf{F}}\right), \tag{167}$$

so that the energy balance (152) reads now

$$\mathrm{tr}\left[\left(\frac{\partial W}{\partial \mathbf{F}} - \mathbf{S}\right) \dot{\mathbf{F}}\right] = 0. \tag{168}$$

Since this identity must be true for all motions, we conclude that

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}, \tag{169}$$

where we used the derivative of a scalar W with respect to the second-order tensor $\mathbf{F}$ defined with respect to Cartesian bases in the reference and current configurations, by

$$\frac{\partial W}{\partial \mathbf{F}} = \frac{\partial W}{\partial F_{ji}} \mathbf{E}_i \otimes \mathbf{e}_j, \qquad \left(\frac{\partial W}{\partial \mathbf{F}}\right)_{ij} = \frac{\partial W}{\partial F_{ji}}. \tag{170}$$

Written in terms of the Cauchy stress this identity provides a *constitutive relationship* relating the Cauchy stress to the deformation gradient:

$$\mathbf{T} = J^{-1} \mathbf{F} \frac{\partial W}{\partial \mathbf{F}}. \tag{171}$$

4.4 Internal material constraint

If we consider a material where the possible deformations are constrained during all motions, extra internal material constraints must be satisfied. We consider the case where these constraints take the form $C(\mathbf{F}) = 0$ where $C(\mathbf{F})$ is a smooth scalar function of the deformation gradient. For instance, in the case of an incompressible material, we assume that all deformations must preserve volume, which implies $\det(\mathbf{F}) = 1$. In this case, $C(\mathbf{F}) = \det(\mathbf{F}) - 1$.

A simple way to ensure that a constraint holds is to introduce a Lagrangian multiplier $p = p(\mathbf{X}, t)$ and modify accordingly the energy density $W \to W - pC$ so that Equation (168) reads now

$$\mathrm{tr}\left[\left(\frac{\partial}{\partial \mathbf{F}}(W - pC) - \mathbf{S}\right) \dot{\mathbf{F}}\right] = 0, \tag{172}$$

which leads to

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}} - p\frac{\partial C}{\partial \mathbf{F}}. \tag{173}$$

In terms of the Cauchy stress, we have

$$\mathbf{T} = J^{-1} \mathbf{F} \mathbf{S} = J^{-1} \mathbf{F} \frac{\partial W}{\partial \mathbf{F}} - p\mathbf{N}, \tag{174}$$

where

$$\mathbf{N} = J^{-1} \mathbf{F} \frac{\partial C}{\partial \mathbf{F}}, \tag{175}$$

is the *reaction stress* enforcing the constraint. In particular, for incompressible materials, we have

$$\frac{\partial C}{\partial \mathbf{F}} = (\det \mathbf{F})\ \mathbf{F}^{-1} = J\mathbf{F}^{-1}, \tag{176}$$

that is, $\mathbf{N} = \mathbf{1}$. The constitutive relationship for an incompressible hyperelastic material is then

$$\mathbf{T} = \mathbf{F}\frac{\partial W}{\partial \mathbf{F}} - p\mathbf{1}. \tag{177}$$

Recalling that a *hydrostatic pressure* is a stress that is multiple of the identity, we can identify the reaction stress in (177) with a hydrostatic pressure. Physically, we see that a pressure p is required to enforce locally the conservation of volume,

For a given $W = W(\mathbf{F})$, the Cauchy stress for compressible or incompressible materials can be written in the general form

$$\mathbf{T} = J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}} - p\mathbf{1}, \tag{178}$$

where $J = 1$ for an incompressible material and $p = p(\mathbf{x}, t)$ must be determined. If the material is unconstrained, then $p = 0$.

5 Summary of equations

■ **Overview** A brief pause to collect all the equations we have seen so far. Counting to make sure that the numbers of unknowns match the muber of equations is always a good idea

We can now collect the different equations from the previous sections to obtain a closed set of equations

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0, \qquad \text{continuity equation} \tag{179}$$

$$\operatorname{div} \mathbf{T} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}, \qquad \text{equation of motion} \tag{180}$$

$$\mathbf{T}^{\mathsf{T}} = \mathbf{T}, \qquad \text{symmetry of Cauchy stress tensor} \tag{181}$$

$$\mathbf{T} = J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}} - p\mathbf{1}, \qquad \text{constitutive law} \tag{182}$$

Since the elements of $\mathbf{F}$ are related to the motion $\boldsymbol{\chi}$ by $\mathbf{F} = \text{Grad}\boldsymbol{\chi}$, and $\mathbf{v} = \partial_t \boldsymbol{\chi}(\mathbf{X}(\mathbf{x}, t), t)$, there are ten unknowns in this system: the scalar field ρ, the vector field $\boldsymbol{\chi}$ and the six components of the symmetric tensor $\mathbf{T}$ for ten equations (excluding the third equation that reduces the number of unknowns in $\mathbf{T}$).

It is also sometimes convenient to write these equations with respect to the reference variables [118]:

$$\dot{\rho}_0 = 0, \qquad \text{continuity equation} \tag{183}$$

$$\text{Div } \mathbf{S} + \rho_0 \mathbf{B} = \rho_0 \dot{\mathbf{v}} \qquad \text{equation of motion} \tag{184}$$

$$\mathbf{S}^{\mathsf{T}}\mathbf{F}^{\mathsf{T}} = \mathbf{F}\mathbf{S}, \qquad \text{symmetry of Cauchy stress tensor} \tag{185}$$

$$\mathbf{S} = \frac{\partial \mathbf{W}}{\partial \mathbf{F}} - p\mathbf{J}\mathbf{F}^{-1}, \qquad \text{constitutive law} \tag{186}$$

where the divergence and gradient are now taken in the initial reference configuration, $\rho_0 = J(\mathbf{X}, t)\rho(\mathbf{x}(\mathbf{X}, t), t)$ is the reference density at a material point, $\mathbf{B} = \mathbf{b}(\mathbf{x}(\mathbf{X}, \mathbf{t}), \mathbf{t})$ is the body force acting at the same point, and $\dot{\mathbf{v}} = \dot{\mathbf{v}}(\mathbf{x}(\mathbf{X}, t), t)$ is the acceleration of a material point.

5.1 Boundary conditions

Equilibrium and static solutions are obtained by setting $\mathbf{v}(\mathbf{X}, t) = 0$ for all $\mathbf{X} \in \mathcal{B}_0$ and for all time t. The equilibrium solutions must satisfy the conditions imposed on the boundary. Depending on the setting, many different types of boundary conditions can be applied and it is well appreciated that the solutions will depend crucially on these conditions. The two main types of boundary conditions are dead loading and rigid loading.

In *dead loading*, a traction is prescribed and maintained constant throughout the deformation, i.e. the prescribed traction is independent of deformation. A typical example of dead loading is *hydrostatic loading* where a constant pressure $P > 0$ is applied at the boundary in the current configuration. In this case, $\mathbf{t}_\mathrm{n} = -P\mathbf{n}$, so that $\mathbf{T} = -P\mathbf{1}$.

In *rigid loading*, fixed displacements are prescribed at the boundary.

In *mixed-loading*, a surface traction $\mathbf{t}_\mathrm{b}$ and deformation $\mathbf{x}_\mathrm{b}$ are prescribed at the boundary:

$$\mathbf{T}\mathbf{n} = \mathbf{t}_\mathrm{b} \qquad \text{for } \mathbf{X} \in \partial\mathcal{B}_0^t \tag{187}$$

$$\mathbf{x}(\mathbf{X}) = \mathbf{x}_\mathrm{b} \qquad \text{for } \mathbf{X} \in \partial\mathcal{B}_0^d \tag{188}$$

where $\partial\mathcal{B}_0^\mathrm{t}$ and $\partial\mathcal{B}_0^\mathrm{d}$ are parts of the body boundary such that $\partial\mathcal{B}_0^\mathrm{t} \cup \partial\mathcal{B}_0^\mathrm{d} = \partial\mathcal{B}_0$ and $\partial\mathcal{B}_0^\mathrm{t} \cap \partial\mathcal{B}_0^\mathrm{d} = \emptyset$. Note that $\mathbf{t}_\mathrm{b}$ can be a function of the deformation gradient as well as of the position.

6 Isotropic materials

■ **Overview** We can make much progress by further restricting our analysis to special claases of materials with given symmetry. The simplest ones are the isotropic materials.

6.1 Objectivity and material symmetry

The functional form of the elastic energy can be restricted by combining a fundamental principle, the principle of objectivity, together with symmetry properties of the material.

The *principle of objectivity* or *material-frame indifference* [146] states that material properties are independent of superimposed rigid-body motions. For hyperelastic materials, the principle of objectivity implies

$$W(\mathbf{QF}) = W(\mathbf{F}), \quad \forall\, \mathbf{Q} \in SO(3), \tag{189}$$

where $SO(3)$ is the set of all proper orthogonal tensors. The principle of objectivity implies that W only depends on $\mathbf{F}$ through $\mathbf{C}$, so that we can write $W(\mathbf{F}) = \bar{W}(\mathbf{C})$. Here, to simplify the notation we will drop the overbar and simply write $W(\mathbf{F}) = W(\mathbf{C})$.

Next, we consider the implication of material symmetries. A material is said to be *symmetric with respect to a linear transformation* if the reference configuration is mapped by this transformation to another configuration which is mechanically indistinguishable from it. The set of all such linear transformations constitutes a symmetry group $\mathbb{Q} \subseteq SO(3)$. The symmetry condition for a hyperelastic material is [118]

$$W(\mathbf{FQ}) = W(\mathbf{F}), \quad \forall\, \mathbf{Q} \in \mathbb{Q}. \tag{190}$$

6.2 symmetry of isotropic materials

The maximal possible symmetry group is $SO(3)$, which defines an *isotropic material*. The case of isotropic materials is particularly important both for its simplicity and its wide applicability. Isotropy implies that the strain-energy function depends on $\mathbf{F}$ only through $\mathbf{V}$, where $\mathbf{V}$ is the symmetric second-order tensor appearing in the polar decomposition $\mathbf{F} = \mathbf{VR}$. Indeed, choosing $\mathbf{Q} = \mathbf{R}^{\mathsf{T}}$ in (190) leads to

$$W(\mathbf{FQ}) = W(\mathbf{FR}^{\mathsf{T}}) = W(\mathbf{VRR}^{\mathsf{T}}) = W(\mathbf{V}). \tag{191}$$

Combining isotropy with objectivity, we have

$$W(\mathbf{QF}\tilde{\mathbf{Q}}^{\mathsf{T}}) = W(\mathbf{F}\tilde{\mathbf{Q}}^{\mathsf{T}}) = W(\mathbf{F}) = W(\mathbf{V}), \quad \forall\, \mathbf{Q}, \tilde{\mathbf{Q}} \in SO(3), \tag{192}$$

and choosing $\tilde{\mathbf{Q}} = \mathbf{QR}$ in $W(\mathbf{QF}\tilde{\mathbf{Q}}^{\mathsf{T}})$ leads to the conclusion that the strain-energy function satisfies

$$W(\mathbf{V}) = W(\mathbf{QVQ}^{\mathsf{T}}), \quad \forall\, \mathbf{Q} \in SO(3). \tag{193}$$

The property (193) defines W as an *isotropic function* of $\mathbf{V}$. By inspection, it can readily be observed that the determinant and trace are simple examples of isotropic functions of a second-order tensor. More generally, an isotropic function of $\mathbf{V}$ can only depend on $\mathbf{V}$ through its three principal invariants

$$\{\mathrm{tr}(\mathbf{V}), \frac{1}{2}\left(\mathrm{tr}(\mathbf{V})^2 - \mathrm{tr}(\mathbf{V}^2)\right), \det(\mathbf{V})\}. \tag{194}$$

However, since $\mathbf{V}$ is a symmetric positive-definite tensor, it is often more convenient to express W through the principal invariants of the left Cauchy-Green tensor $\mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^\mathsf{T}$:

$$I_1 = \mathrm{tr}(\mathbf{B}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \tag{195}$$

$$I_2 = \frac{1}{2}\left(I_1^2 - \mathrm{tr}(\mathbf{B}^2)\right) = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2, \tag{196}$$

$$I_3 = \det(\mathbf{B}) = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{197}$$

Equivalently, it implies that W only depends on $\mathbf{F}$ through its principal stretches λ_1, λ_2, λ_3 (the square roots of the principal values of $\mathbf{B}$). With a slight abuse of notation, we write either $W = W(I_1, I_2, I_3)$, or $W = W(\lambda_1, \lambda_2, \lambda_3)$.

For an isotropic compressible material, we have $J = 1$, which implies that $\lambda_3 = 1/(\lambda_1\lambda_2)$. Therefore, W can be either expressed in terms of $\{\lambda_1, \lambda_2\}$ or $\{I_1, I_2\}$.

The explicit form of the Cauchy stress tensor for a compressible material in terms of the invariants and their derivatives is

$$\mathbf{T} = w_0\mathbf{1} + w_1\mathbf{B} + w_2\mathbf{B}^2, \tag{198}$$

where the functions w_i depend on the invariants and are given explicitly by

$$w_0 = 2J\frac{\partial W}{\partial I_3} - p, \tag{199}$$

$$w_1 = 2J^{-1}\frac{\partial W}{\partial I_1} + 2J^{-1}\frac{\partial W}{\partial I_2}I_1, \tag{200}$$

$$w_2 = -2J^{-1}\frac{\partial W}{\partial I_2}. \tag{201}$$

As before we choose $p = 0$ for compressible materials and $J = I_3 = 1$ for incompressible materials. If the reference configuration is assumed stress-free, then we must have $\mathbf{T}(\mathbf{F} = \mathbf{1}) = \mathbf{0}$, that is the functions $w_i = w_i(I_1, I_2, I_3)$ satisfy

$$w_0(3,3,1) + w_1(3,3,1) + w_2(3,3,1) = 0. \tag{202}$$

A convenient alternative representation, the *Rivlin–Ericksen representation*, is obtained from (198) by using Cayley–Hamilton's theorem [60] for $\mathbf{B}$. In terms of the invariants, the Cayley–Hamilton theorem in three dimensions reads

$$\mathbf{B}^3 - I_1\mathbf{B}^2 + I_2\mathbf{B} - I_3\mathbf{1} = 0. \tag{203}$$

Substituting $\mathbf{B}^2 = I_1\mathbf{B} - I_2\mathbf{1} + I_3\mathbf{B}^{-1}$ in (198) gives

$$\mathbf{T} = \beta_0\mathbf{1} + \beta_1\mathbf{B} + \beta_{-1}\mathbf{B}^{-1}, \tag{204}$$

where

$$\beta_0 = 2J\frac{\partial W}{\partial I_3} + 2J^{-1}I_2\frac{\partial W}{\partial I_2} - p, \tag{205}$$

$$\beta_1 = 2J^{-1}\frac{\partial W}{\partial I_1}, \tag{206}$$

$$\beta_{-1} = -2J\frac{\partial W}{\partial I_2}. \tag{207}$$

6.3 Adscititious inequalities

The principle of objectivity together with isotropy leads to a representation of the strain-energy function in terms of three invariants. This formulation still leads to many choices for a suitable functional form of W. If we want to establish general results independent of the particular choice of strain-energy function, we can impose certain desired behaviors [151]. For instance, we may require that, in simple extension an isotropic elastic body extends rather than shrinks. These conditions take the form of inequalities either on the coefficients of the constitutive relations, or on the principal stresses and strains. They are called *adscititious inequalities* as they come from outside the theory and are derived empirically from everyday experience or experiments [147]. We mention here three standard inequalities often used in elasticity and suitable for most elastomers.

- **Baker-Ericksen inequalities.** The Baker-Ericksen inequalities follow from the requirement that the greater principal stress occurs in the direction of the greater principal stretch [9] which implies

$$\lambda_i \neq \lambda_j \quad \Rightarrow \quad (t_i - t_j)(\lambda_i - \lambda_j) > 0, \qquad i, j = 1, 2, 3, \tag{208}$$

  where $\{t_1, t_2, t_3\}$ and $\{\lambda_1, \lambda_2, \lambda_3\}$ are the principal stresses and principal stretches obtained by the spectral decomposition

$$\mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i, \qquad \mathbf{T} = \sum_{i=1}^{3} t_i \mathbf{v}_i \otimes \mathbf{v}_i. \tag{209}$$

  Condition (208) imposes the following restrictions on the coefficients in (204):

$$\lambda_i^2 \lambda_j^2 \beta_1 > \beta_{-1}, \quad \text{if } \lambda_i \neq \lambda_j, \tag{210}$$

$$\lambda_i^4 \beta_1 \geqslant \beta_{-1}, \qquad \text{if } \lambda_i = \lambda_j. \tag{211}$$

  For a hyperelastic body under uniaxial tension, the deformation is a simple extension in the direction of the (positive) tensile force. The ratio of the tensile strain to the strain in the perpendicular direction is greater than one if and only if the Baker-Ericksen inequalities hold [103].

- **The ordered-forces inequalities.** Similar to the Baker-Ericksen inequalities, the *ordered-forces inequalities* state that the greater stretch occurs in the direction of the greater force. While similar, the two sets of inequalities do not imply each other [146, p. 157]. However, it can be shown that if two of the three principal stresses are non-negative, then the Baker-Ericksen inequalities follow from the ordered-forces inequalities, and if two of the three principal stresses are non-positive, then the ordered-forces inequalities are implied by the Baker-Ericksen inequalities.

- **Empirical inequalities.** Based on experimental observations in elastomers, the following *empirical inequalities* on the coefficients of (204) have been postulated [115]:

$$\beta_0 \leqslant 0, \qquad \beta_1 > 0, \qquad \beta_{-1} \leqslant 0, \tag{212}$$

  for the compressible case. For the incompressible case, only the last two inequalities are considered. These inequalities directly imply the Baker-Ericksen inequalities (210) but not conversely.

6.3.1 Example: pure shear of an elastic cube

The adscititious inequalities can be used to establish general qualitative trends. For instance, consider a homogeneous isotropic hyperelastic cube subject to a *pure shear stress* on its top face as shown in Figure 10. In Cartesian coordinates, this stress can be written as $\mathbf{T} = T(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$, where $T > 0$ is constant. Equivalently, in matrix form, it reads

$$[T] = \begin{bmatrix} 0 & T & 0 \\ T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{213}$$

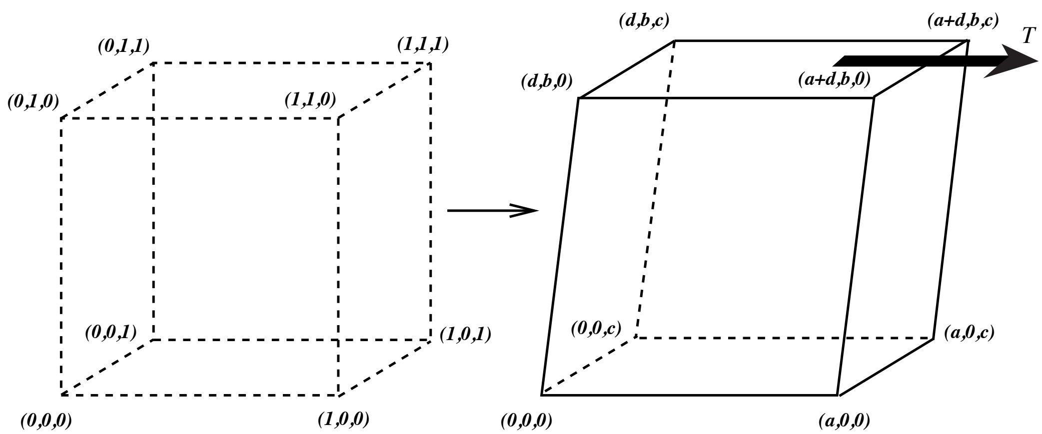


Figure 10: A cube deformed under pure shear stress T applied to the top face ($d = \sqrt{b^2 - a^2}$).

Since T is constant, in the absence of body force, the equation of motion is identically satisfied. The corresponding deformation is homogeneous. That is the deformation gradient is independent of the position. For a pure shear stress, it reads

$$x = aX + \sqrt{b^2 - a^2}Y, \qquad y = bY, \qquad z = cZ. \tag{214}$$

It consists of a triaxial stretch, a pure strain deformation, combined with a simple shear in the *direction* of the shear force *if and only if* the Baker-Ericksen inequalities hold. Therefore, the Baker-Ericksen inequalities guarantee that the shear strain is in the same direction as the shear force.

If the Baker-Ericksen inequalities are not satisfied, the material would shear in the direction opposite to the direction of the shear stress. This behavior would be unrealistic, even though there is no fundamental principle that would rule it out. It is only through our own experience of everyday materials that we infer its physical impossibility.

For the deformation (214), obtained under pure shear, we define the *Poynting effect* as $0 < b \neq 1$ [115]. Specifically, if a cube is deformed under pure shear, the *positive Poynting effect* occurs when $b > 1$ [13], i.e. the sheared faces spread apart, whereas the *negative Poynting effect* is obtained when $b < 1$, i.e. the sheared faces are drawn together. It can be shown that the validity of the empirical inequalities (with $\beta_{-1} < 0$) is a necessary and sufficient condition for the positive Poynting effect as found in rubber materials [113]. However, experimental observations suggests that semi-flexible polymer gels exhibit a *negative* Poynting effect [84], which implies $\beta_{-1} > 0$. Therefore, it appears that the last of the empirical inequalities may not hold for some biological materials and should be replaced by the *generalized empirical inequalities* [114] which simply state $\beta_0 \leqslant 0$ and $\beta_1 > 0$.

6.4 Choice of strain-energy functions

The choice of strain-energy functions $W = W(\mathbf{F})$ for particular applications is a controversial and difficult problem. Methods based on a statistical analysis of the microstructure have been proposed [16, 39]. However, typically, phenomenological models are used to capture the essential features of a material such as its behavior under shear or its strain-hardening or strain-softening properties [76, 79, 134], while respecting basic material properties such as convexity and objectivity [154]. Note that the words *model* and *material* are used exchangeably to describe these particular strain-energy functions. For instance, a neo-Hookean material is a material described by the neo-Hookean model, that is a hyperelastic material with the particular form of strain-energy density function given below. These models can be calibrated and fitted to uniaxial or biaxial experiments [72, 73, 92, 96, 135, 136, 150, 148]. Here, we limit our presentation to a few key popular models for incompressible materials that capture specific features and are widely used in applications. A summery is given in Table 1.

- **Neo-Hookean materials.** The simplest model, and the starting point of many theories, is the *neo-Hookean model* [40]:

$$W_{\text{nh}} = \frac{C_1}{2}(I_1 - 3). \tag{215}$$

  This strain-energy function can be derived from statistical mechanics as a macroscopic limit of the energy density of an amorphous cross-linked network of polymeric molecules [38, 39, 143, 144]. Each molecule in this network is a freely jointed chain with the same number of monomer units and their end-to-end distances follow a Gaussian distribution.

  For small deformations, the macroscopic parameter C_1 can be identified with the shear modulus μ and is proportional to the product $k_{\text{B}}T$ of the Boltzmann constant with the absolute temperature. The Young's modulus is then related to C_1 by

$$E = 3\mu = 3C_1. \tag{216}$$

  A simple generalization of neo-Hookean material is obtained by assuming that the strain-energy function is only a function of the first invariant, $W(\mathbf{F}) = W(I_1)$. This class of models are know as *generalized neo-Hookean* materials [4, 155], for which general results can easily be established [100].

- **Mooney-Rivlin materials.** The neo-Hookean model can be interpreted as the lowest approximation of a strain-energy function with respect to the strain tensor $\mathbf{E} = (\mathbf{F}^{\mathsf{T}}\mathbf{F} - \mathbf{1})/2$, the so-called *second-order elasticity* approximation. This model is a good descriptor for many elastomers in tension or compression but it often fails to capture quantitatively behaviors associated with shear or torsion. The next order approximation for incompressible isotropic elasticity gives rise to third-order elasticity [27, 32, 58]. Expressed in terms of invariants, it takes the form of the Mooney-Rivlin strain-energy function

$$W_{\text{mr}} = \frac{C_1}{2}(I_1 - 3) + \frac{C_2}{2}(I_2 - 3). \tag{217}$$

  The combination $C_1 + C_2 = \mu$ can be identified again as the shear modulus. Therefore, we can write

$$C_1 = \mu(\frac{1}{2} + \alpha), \quad C_2 = \mu(\frac{1}{2} - \alpha). \tag{218}$$

The Baker-Ericksen inequalities imply that $\alpha \in [-1/2, 1/2]$. This model can also be derived from statistical-mechanics arguments by relaxing some of the assumptions that lead to the neo-Hookean model [45].

- **Ogden materials.** A general approach for material modeling, originally proposed by Ogden [117, 119, 121], consists in considering a general expansion with N terms of the form

$$W_{\mathrm{og}N} = \sum_{i=1}^{N} \frac{\mu_i}{\alpha_i} \left(\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3\right). \tag{219}$$

Each parameter μ_i and α_i is a material constant to be determined. These constants are related to the shear modulus μ of small deformations by

$$\sum_{i=1}^{N} \mu_i \alpha_i = 2\mu. \tag{220}$$

In practice, the number of terms is limited to $N \leqslant 6$. The possibility of having a large number of parameters provides a systematic way to explore many different behaviors and can be used to fit experimental data of both elastomers and biological tissues [82, 112, 152, 153].

- **Fung-Demiray materials.** Many soft tissues and elastomers exhibit strong strain-hardening properties. That is, in simple extension, it becomes increasingly difficult to further extend the material. Examples of generalized neo-Hookean materials that have been used to capture this effect in soft tissues are the Fung and Gent models. The *Fung model* [49, 50, 94, 142], in its simplest form, reads

$$W_{\mathrm{fu}} = \frac{\mu}{2\beta}[\exp \beta(I_1 - 3) - 1], \tag{221}$$

where $\beta > 0$ controls the strain-hardening property. In the limit $\beta \to 0$ the Fung model reduces to the standard neo-Hookean model. This particular form of the Fung model was first proposed by Demiray in 1972 [29, 30].

- **Gent materials.** Another popular model is the *Gent model* that has finite-chain extensibility enforced by a singular limit of the strain-energy function [53, 77, 79]

$$W_{\mathrm{ge}} = -\frac{\mu}{2\beta} \log[1 - \beta(I_1 - 3)]. \tag{222}$$

The neo-Hookean model is obtained in the limit $\beta \to 2$.

For a compressible material, there is an extra dependance on the invariant $I_3 = J^2$. Again, a choice must be made to take into account the energy associated with local changes of the volume. Typically, an additive choice is made for which the strain-energy function is separated into an incompressible part and a compressible part [74, 101] so that

$$W = W_{\mathrm{inc}}(I_1, I_2) + W_{\mathrm{comp}}(I_3) \tag{223}$$

and a model from Table 1 is adopted for $W_{\mathrm{inc}}(I_1, I_2)$. Possible choices for $W_{\mathrm{comp}}(I_3)$ include: $\mu_{\mathrm{c}}(I_3 - 1)$, $\mu_{\mathrm{c}}(J-1)^2$, $\mu_{\mathrm{c}} \ln I_3$, $\mu_{\mathrm{c}} \ln J$, where μ_{c} is a material parameter related to the bulk modulus.

Name	Definition	soft tissues	elastomers
neo-Hookean	$W_{\text{nh}} = \frac{C_1}{2}(I_1 - 3)$		
Mooney-Rivlin	$W_{\text{mr}} = \frac{C_1}{2}(I_1 - 3) + \frac{C_2}{2}(I_2 - 3)$		
Ogden 1	$W_{\text{og1}} = \dfrac{2\mu}{\beta^2}(\lambda_1^\beta + \lambda_2^\beta + \lambda_3^\beta - 3)$	$\beta \geq 9$	$\beta \approx 3$
Fung	$W_{\text{fu}} = \dfrac{\mu}{2\beta}[\exp \beta(I_1 - 3) - 1]$	$3 < \beta < 20$	
Gent	$W_{\text{ge}} = -\dfrac{\mu}{2\beta} \log[1 - \beta(I_1 - 3)]$	$0.4 < \beta < 3$	$0.005 < \beta < 0.05$

Table 1: A list of phenomenological strain-energy functions for isotropic incompressible materials. Note that the materials have been written so that they share the same infinitesimal shear modulus μ. The limits $\beta \to 2$ in W_{og1} and, $\beta \to 0$ in W_{fu} and W_{ge} all lead to the neo-Hookean strain-energy function. Estimates are taken from: 1-term Ogden is [15, 138], Gent [53, 55, 76, 78], Fung [28, 71].

7 Examples of boundary value problems

■ **Overview** For a given strain-energy density function, we can write a full system of equations which can be solved for given boundary conditions. We (finally!) give some simple solutions for homogeneous and semi-inverse problems.

7.1 A simple homogeneous deformation

Homogeneous deformations are specified by a constant deformation gradient so that

$$\mathbf{x} = \mathbf{F}\mathbf{X} + \mathbf{c}, \tag{224}$$

where $\mathbf{F}$ and $\mathbf{c}$ are constant.

For a homogenous isotropic compressible material, this choice implies that $\mathbf{T}$ is constant, and in the absence of a body load, the equilibrium equations are therefore trivially satisfied. The solution is then fully specified by the boundary conditions and the constitutive law [133].

Consider for example, the diagonal deformation of a cuboid into another cuboid shown in Figure 11 and described by

$$x_i = \lambda_i X_i, \qquad i = 1, 2, 3 \text{ (no summation over } i). \tag{225}$$

The corresponding deformation gradient is $[\mathbf{F}] = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ and, consequently, the

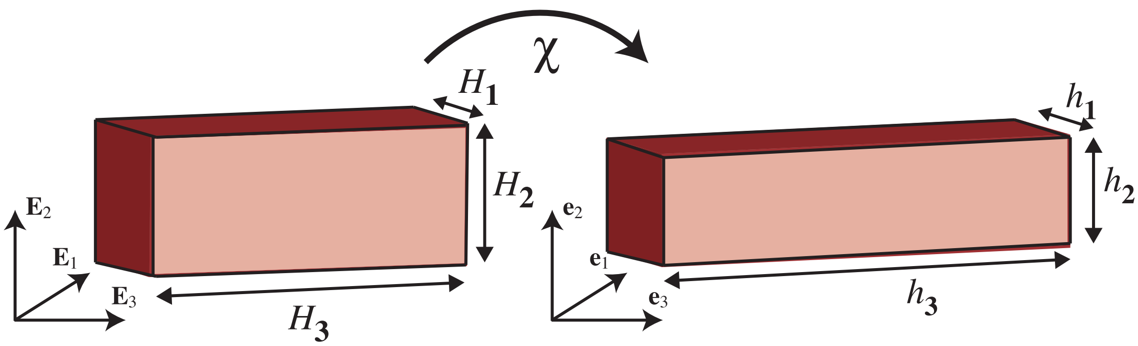


Figure 11: A simple diagonal homogenous deformation transforming a cuboid into another cuboid. The deformation is characterized by the three constants $\lambda_i = h_i/H_i,\ i = 1, 2, 3$.

Cauchy stress tensor is also diagonal: $[\mathbf{T}] = \text{diag}(t_1, t_2, t_3)$. The deformation and stresses are then obtained from the constitutive law

$$t_i = \frac{\lambda_i}{J}\frac{\partial W}{\partial \lambda_i}, \qquad i = 1, 2, 3 \text{ (no summation over } i). \tag{226}$$

For instance, in a hydrostatic uniaxial extension, the stress is prescribed in one direction, that is $t_3 = N$ and $t_1 = t_2 = 0$. By symmetry, we have $\lambda_1 = \lambda_2$ and

$$t_2 = \frac{1}{\lambda_2\lambda_3}W_2 = 0, \quad t_3 = \frac{1}{\lambda_2^2}W_3 = N. \tag{227}$$

These equations constitute a systems of two equations for two unknowns.

We further specialize these equations to a compressible neo-Hookean material with a strain-energy function of the form

$$W = \frac{\mu_1}{2}(I_1 - 3) - \frac{\mu_1}{2}(I_3 - 1) + \frac{\mu_2}{4}(I_3 - 1)^2. \tag{228}$$

This particular form is chosen so that all stresses vanish at $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Equations (227) then read

$$0 = \mu_1(\frac{1}{\lambda_3} - \lambda_2^2\lambda_3) + \mu_2\lambda_2^2\lambda_3(\lambda_2^4\lambda_3^2 - 1), \tag{229}$$

$$N = \mu_1(\frac{\lambda_3}{\lambda_2^2} - \lambda_2^2\lambda_3) + \mu_2\lambda_2^2\lambda_3(\lambda_2^4\lambda_3^2 - 1). \tag{230}$$

The Young's modulus E is obtained for small deformations as the ratio of uniaxial stress to the uniaxial stretch, that is

$$\begin{aligned} E &= \left.\frac{\partial N(\lambda_2,\lambda_3)}{\partial \lambda_3}\right|_{\lambda_2=\lambda_3=1} + \left.\left(\frac{\partial N(\lambda_2,\lambda_3)}{\partial \lambda_2}\frac{\partial \lambda_2}{\partial \lambda_3}\right)\right|_{\lambda_2=\lambda_3=1} \\ &= 2\mu_1\frac{2\mu_1 - 3\mu_2}{\mu_1 - 2\mu_2}. \end{aligned} \tag{231}$$

If we now consider the same deformation but for an incompressible material with a neo-Hookean strain-energy function $W = \mu(I_1 - 3)/2$, we have $\lambda_1 = \lambda_2$ again by symmetry, but $\lambda_3\lambda_2^2 = 1$ by incompressibility. We replace the constitutive law (226) by

$$t_i = \lambda_i\frac{\partial W}{\partial \lambda_i} - p \qquad i = 1, 2, 3 \text{ (no summation over } i). \tag{232}$$

The boundary conditions lead to

$$p = \frac{\mu}{\lambda_3}, \qquad N(\lambda_3) = \frac{\left(\lambda_3^3 - 1\right)\mu}{\lambda_3}, \tag{233}$$

which defines a Young's modulus

$$E = \left.\frac{\partial N(\lambda_3)}{\partial \lambda_3}\right|_{\lambda_3=1} = 3\mu. \tag{234}$$

7.2 The half-plane in compression

As a second example, we consider another type of homogeneous deformation applied to an incompressible hyperelastic half-space with a free surface, characterized by $W = W(\lambda_1, \lambda_2, \lambda_3)$, under pure homogeneous static deformation with principal stretch ratios $\lambda_1, \lambda_2, \lambda_3$ [14, 59]. We take $\mathbf{e}_2 = \mathbf{E}_2$ normal to the surface with the half-space located in $X_2 > 0$, so that λ_2 is the stretch ratio in the direction normal to the free surface. We consider homogeneous loadings so that the Cauchy stress tensor, in Cartesian coordinates, can be written $[\mathbf{T}] = \text{diag}(t_1, t_2, t_3)$ with deformation gradient $[\mathbf{F}] = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$.

There are three typical types of loading which are of interest as shown in Figure 12:

- **Equibiaxial strain.** An *equibiaxial strain* is associated with deformations with equal strains in the plane, that is $\lambda_1 = \lambda_3$. In this case, the half-space is compressed with equal force in the 1- and 3-directions, and expands freely in the 2-direction, so that

$$t_1 = \lambda_1 W_1 - \lambda_2 W_2, \quad t_2 = 0, \quad t_3 = t_1, \tag{235}$$

where $W_i = \partial_{\lambda_i} W$.

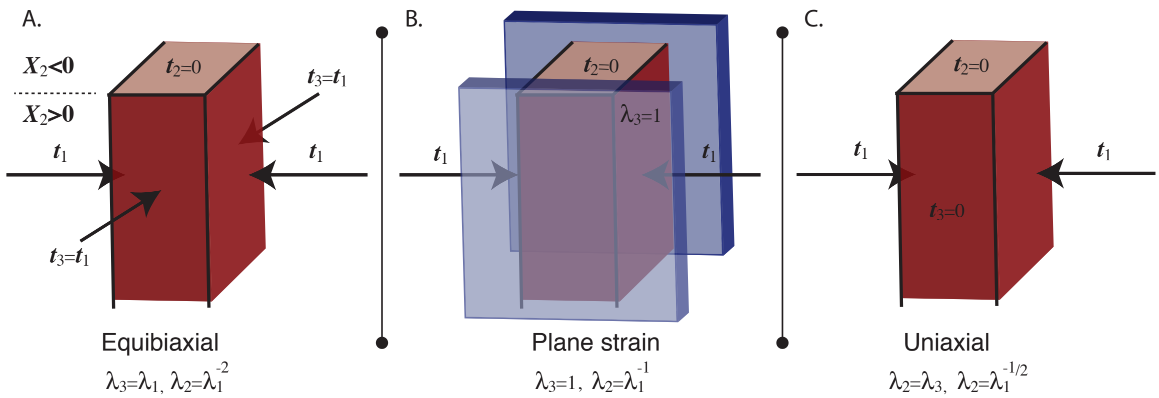


Figure 12: Three typical deformations of a half space. A. Equibiaxial strain. B. Plane strain. C. Uniaxial strain

- **Plane strain.** *Plane strain* corresponds to the condition $\lambda_3 = 1$. It corresponds to a half-space compressed in the 1-direction and prevented from expanding/contracting in the 3-direction. Thus, it expands in the 2-direction, normal to the free surface. This deformation is maintained by applying the tractions

$$t_1 = \lambda_1 W_1 - \lambda_2 W_2, \quad t_2 = 0, \quad t_3 = \lambda_3 W_3 - \lambda_2 W_2 \neq t_1. \tag{236}$$

- **Uniaxial strain.** A *uniaxial strain* is defined by $t_3 = 0$. That is, the half-space is free to expand in the 2- and 3-directions, and

$$t_1 = \lambda_1 W_1 - \lambda_2 W_2, \quad t_2 = 0, \quad t_3 = 0. \tag{237}$$

It follows from the incompressibility condition that $\lambda_1\lambda_2\lambda_3 = 1$ and therefore the three cases can be written in general as

$$\lambda_2 = \lambda_1^n \quad \text{with} \quad \begin{cases} n = -2 & \text{equibiaxial,} \\ n = -1 & \text{plane strain,} \\ n = -\frac{1}{2} & \text{uniaxial.} \end{cases} \tag{238}$$

Therefore, we can use λ_1 to fully characterize the deformation. For a given strain-energy function, the stresses developed as a function of λ_1 can be computed by direct evaluation of the relations (235–236).

Once the stresses are known, a natural problem is to look for possible bifurcations. That is, we wish to identify a critical value of λ_1 such that the half-space develops surface wrinkles as one would expect when compressing a large rubber block. We will not consider this problem here but it is an active area of research.

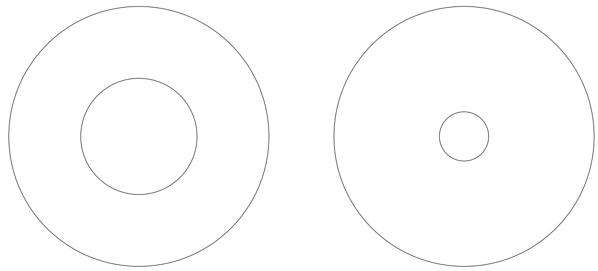
7.3 Inflation of a spherical shell.

1) Elastic, incompressible, isotropic spherical shell with strain-energy $W(I_1, I_2, I_3)$.

2) Symmetric inflation

$$A \leqslant R \leqslant R, \qquad \mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \quad \text{(see §2.9)} \tag{239}$$

$$\mathbf{x} = f(R)\mathbf{X}, \qquad r = f(R)R. \tag{240}$$



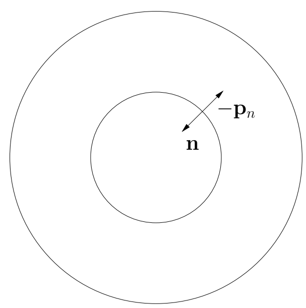
$$\mathbf{F} = \begin{bmatrix} \lambda_r & & \\ & \lambda_\theta & \\ & & \lambda_\phi \end{bmatrix} = \begin{bmatrix} r' & & \\ & r/R & \\ & & r/R \end{bmatrix}. \tag{241}$$

$$\lambda_r = r'(R), \ \lambda_\theta = r/R = \lambda_\phi \tag{242}$$

$$\lambda_a = a/A, \ \lambda_b = b/B, \ r = \sqrt[3]{a^3 - A^3 + R^3} \tag{243}$$

where a is the single unknown parameter. Therefore

$$\lambda_\theta = \lambda_\phi = \lambda = r/R, \qquad \lambda_r = \lambda^{-2} \tag{244}$$



$$\mathbf{T} \cdot \mathbf{n} = \begin{cases} -P & \text{on } \ r = a \\ 0 & \text{on } \ r = b \end{cases} \tag{245}$$

$\mathbf{S}^T \cdot \mathbf{N} = JP\mathbf{F}^{-T}\mathbf{N}$ (mapping of traction vector)

$$\begin{cases} \mathbf{Tn} = -P\mathbf{n} & \text{on } \ \partial\mathcal{B} \\ \mathbf{S}^T \cdot \mathbf{N} = -PJ\mathbf{F}^{-T}\mathbf{N}, & \text{on } \ \partial\mathcal{B} \end{cases} \tag{246}$$

$$\implies T_{rr} = \begin{cases} -P & \text{on } \ r = a \\ 0 & \text{on } \ r = b \end{cases} \tag{247}$$

or

$$S_{rr} = \begin{cases} -P\lambda_r^{-1} = -P\lambda^2 & \text{on } \ R = A \\ 0 & \text{on } \ R = B \end{cases} \tag{248}$$

Note that the boundary condition depends on the deformation.

3) $\mathbf{b} = 0$ and div $\mathbf{T} = 0$,

$$\implies \frac{\mathrm{d}T_{rr}}{\mathrm{d}r} + \frac{2}{r}\left(T_{rr} - T_{\theta\theta}\right) = 0, \tag{249}$$

or

$$\frac{\mathrm{d}S_{rr}}{\mathrm{d}R} + \frac{2}{R}\left(S_{rr} - S_{\theta\theta}\right) = 0. \tag{250}$$

Constitutive equations,

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}} - p\mathbf{F}^{-1}, \tag{251}$$

or

$$\mathbf{T} = \mathbf{F}\frac{\partial W}{\partial \mathbf{F}} - p\mathbb{1}. \tag{252}$$

Then

$$S_{rr} = \frac{\partial W}{\partial \lambda_r} - p\lambda_r^{-1}, \qquad S_{\theta\theta} = \frac{\partial W}{\partial \lambda_\theta} - p\lambda_\theta^{-1} \tag{253}$$

which are functions of $\lambda(R)$.

4) Solve the equation div $\mathbf{S} = 0$. To do so, we choose λ as a variable. Define $h(\lambda) = W(\lambda^{-2}, \lambda, \lambda)$, then

$$\frac{\mathrm{d}S_{rr}}{\mathrm{d}\lambda} = -2\frac{S_{rr} - S_{\theta\theta}}{\lambda - \lambda^{-2}} = -\frac{h'(\lambda)}{\lambda^3 - 1},$$

$$\implies \boxed{P = \int_{\lambda_b}^{\lambda_a} \frac{h'(\lambda)}{\lambda^3 - 1}}$$

$$\frac{\partial t_r}{\partial r} + \frac{2}{r}(t_r - t_\theta) = 0, \tag{254}$$

$$t_r = \lambda_r W_r - p = \lambda^{-2} W_r - p, \qquad t_\theta = \lambda W_\theta - p, \qquad t_\phi = t_\theta. \tag{255}$$

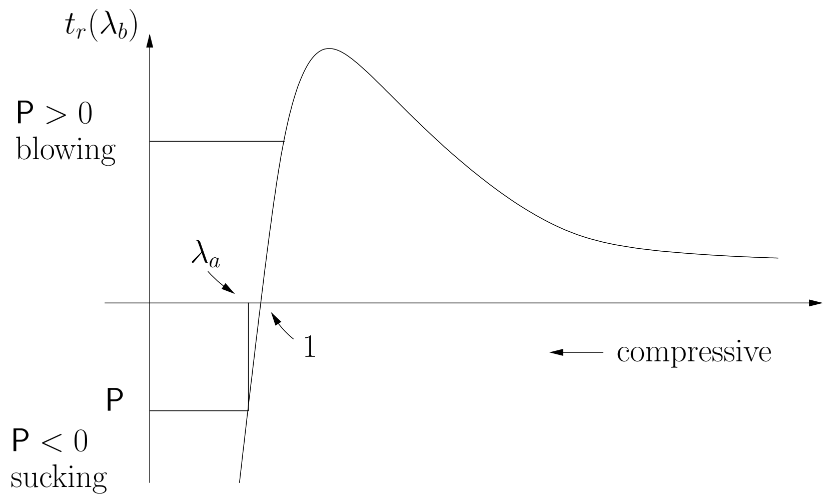
$$t_r - t_\theta = \lambda^{-2} W_r - \lambda W_\theta \tag{256}$$

$$\implies \frac{\partial t_r}{\partial r} + \frac{2}{r}(\lambda^{-2} W_r - \lambda W_\theta) = 0. \tag{257}$$

Introduce auxiliary function, $h(\lambda) = W(\lambda^{-2}, \lambda, \lambda)$,

$$h'(\lambda) = \frac{\partial h}{\partial \lambda} = W_r.(-2\lambda^{-3}) + W_\theta.1 + W_\phi.1 = -2\lambda^{-1}(\lambda^{-2} W_r - \lambda W_\theta) \tag{258}$$

$$\frac{\partial t_r}{\partial r} = \frac{\partial t_r}{\partial \lambda}\frac{\partial \lambda}{\partial r} \tag{259}$$



$$\lambda = \frac{r}{R(r)}, \qquad \frac{\partial \lambda}{\partial r} = \frac{1}{R} - \frac{rR'}{R^2}$$

$$R^3 = r^3 - a^3 + A^3, \qquad R'R^2 = 3r^3, \qquad R' = \frac{r^2}{R^2} = \lambda^2.$$

$$\frac{\partial \lambda}{\partial r} = \frac{1}{R}(1 - \lambda^3) \tag{260}$$

$$\implies \frac{\partial t_r}{\partial r} = \frac{\partial t_r}{\partial \lambda}\frac{1}{R}(1-\lambda^3) = \frac{\lambda h'(\lambda)}{r}. \tag{261}$$

$$\frac{\partial t_r}{\partial \lambda} = \frac{h'(\lambda)}{1-\lambda^3}, \qquad \implies t_r = \int_{\lambda_a}^{\lambda} \frac{h'(\lambda)}{1-\lambda^3}\,\mathrm{d}\lambda \tag{262}$$

At $\lambda = \lambda_b$, $t_r = -P$,

$$-P = -\int_{\lambda_a}^{\lambda_b} \frac{h'(\lambda)}{1-\lambda^3}\,\mathrm{d}\lambda, \qquad \implies \qquad P = \int_{\lambda_a}^{\lambda_b} \frac{h'(\lambda)}{1-\lambda^3}\,\mathrm{d}\lambda = f(\lambda_a). \tag{263}$$

Note

$$\lambda_a = a/A, \qquad \lambda_b = \frac{1}{B}\sqrt[3]{a^3 - A^3 + B^3} = \frac{1}{B}\sqrt[3]{(\lambda_a - 1)A^3 + B^3}. \tag{264}$$

For a given P, we find a, hence the deformation and the value of t_r at all points. Note that $W = \frac{\mu}{2}(\lambda_r^2 + \lambda_\theta^2 + \lambda_\phi^2)$,

$$\implies h = \frac{\mu}{2}\left(\frac{1}{\lambda^4} + 2\lambda^2\right). \tag{265}$$

Note the nonlinearity in the $1/\lambda^4$ term.

$$\implies \frac{h'}{1-\lambda^3} = -2\mu(\lambda^{-2} + \lambda^{-5}), \tag{266}$$

$$P = -2\mu\left(\frac{1}{\lambda} + \frac{1}{4\lambda^4}\right)\Bigg|_{\lambda_a}^{\lambda_b(\lambda_a)}. \tag{267}$$

7.4 The inflation-extension of a tube

As an example of non-homogeneous static deformations, we consider the problem of an incompressible hyperelastic cylindrical shell subject to combined extension and inflation [34, 80, 88] in the absence of body loads. We consider a simple thought experiment in which the tube is capped at both ends and subject to an axial extension ζ due to an internal pressure P and to a total axial load N on the top cap. The tube of initial inner radius A and outer radius $B > A$, and height H is then deformed into a tube with radii a, b and height h as shown in Figure 13. We consider a finite deformation in which the cylinder is allowed to inflate and extend while remaining cylindrical at all times regardless of possible stability issues [12, 21, 57, 65, 64].

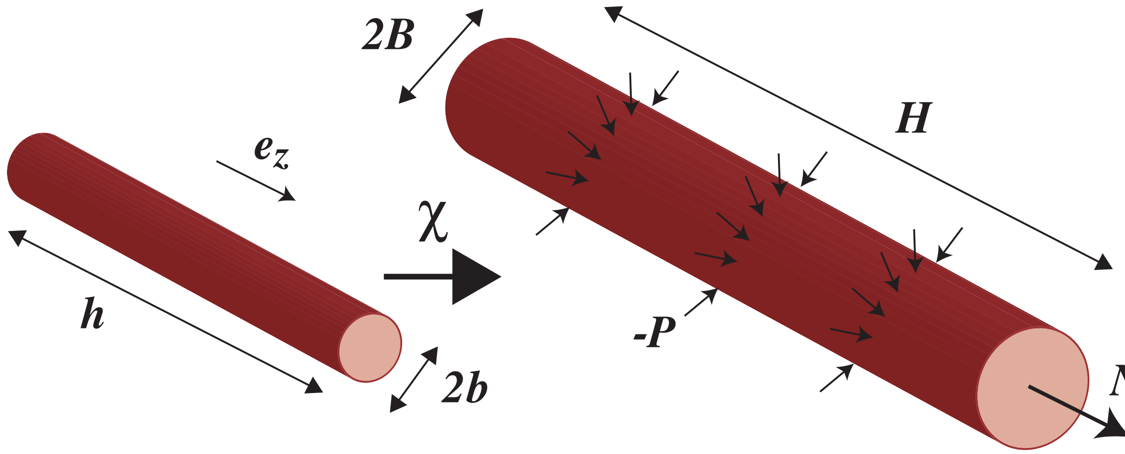


Figure 13: Inflation-extension of a tube. The tube is inflated with an internal pressure P (which is equivalent to an external pressure $-P$ as shown) and an axial load N resulting from an applied load and the pressure acting on the end caps.

For this problem, the deformation $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, in cylindrical coordinates $\{r, \theta, z\}$ and $\{R, \Theta, Z\}$ reads

$$r = r(R), \quad \theta = \Theta, \quad z = \zeta Z, \tag{268}$$

where ζ is the constant axial stretch of the cylinder such that $h = \zeta H$. The position vectors in the reference and current configurations are

$$\mathbf{X} = R\mathbf{E}_R + Z\mathbf{E}_Z, \quad \mathbf{x} = r(R)\mathbf{e}_r + \zeta Z\mathbf{e}_z. \tag{269}$$

where $\{\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ are the two standard cylindrical bases. Following the identity (76), the deformation gradient $\mathbf{F} = \text{Grad}(\boldsymbol{\chi})$ with respect these coordinates is given by

$$\mathbf{F} = r'\mathbf{e}_r \otimes \mathbf{E}_R + \frac{r}{R}\mathbf{e}_\theta \otimes \mathbf{E}_\theta + \zeta\mathbf{e}_z \otimes \mathbf{E}_Z, \tag{270}$$

where the prime denotes differentiation with respect to R. Equivalently, we can write

$$[\mathbf{F}] = \text{diag}(r', \frac{r}{R}, \zeta) \equiv \text{diag}(\lambda_r, \lambda_\theta, \lambda_z), \tag{271}$$

which defines the three principle stretches $\{\lambda_r, \lambda_\theta, \lambda_z\}$.

The incompressibility condition $\det(\mathbf{F}) = 1 = \lambda_r\lambda_\theta\lambda_z$ leads to

$$r'r = \frac{R}{\zeta}, \tag{272}$$

which, together with $r(A) = a$, leads to

$$r = \sqrt{a^2 + \frac{R^2 - A^2}{\zeta}}. \tag{273}$$

Then, $\lambda = \lambda_\theta$ is given by

$$\lambda = \frac{r}{R} = \frac{1}{R}\sqrt{a^2 + \frac{R^2 - A^2}{\zeta}}. \tag{274}$$

Therefore, the deformation is fully specified by two parameters: the axial stretch ζ and the radial stretch of the inner wall $\lambda_a = a/A$ so that

$$\lambda_b = \frac{b}{B} = \frac{1}{\zeta}\sqrt{1 + \frac{A^2}{B^2}(\zeta\lambda_a^2 - 1)}. \tag{275}$$

Since the deformation is diagonal in cylindrical coordinates and only depends on R, it follows from Equation (198) that the Cauchy stress tensor is also diagonal in these coordinates so that

$$[\mathbf{T}] = \text{diag}(t_r, t_\theta, t_z) \;\Leftrightarrow\; \mathbf{T} = t_r\mathbf{e}_r \otimes \mathbf{e}_r + t_\theta\mathbf{e}_\theta \otimes \mathbf{e}_\theta + t_z\mathbf{e}_z \otimes \mathbf{e}_z. \tag{276}$$

This particular form of the Cauchy stress tensor implies that the Cauchy equation $\text{div}\,\mathbf{T} = 0$ in cylindrical coordinates reduces to a single scalar equation

$$\frac{\mathrm{d}t_r}{\mathrm{d}r} + \frac{1}{r}(t_r - t_\theta) = 0. \tag{277}$$

This equation can be integrated once over r:

$$t_r(r) = t_r(a) + \int_a^r \frac{t_\theta - t_r}{r}\mathrm{d}r, \quad r \in [a, b]. \tag{278}$$

We now examine the boundary conditions. First, due to inflation, the jump in pressure between the inner and outer sides of the cylinder is $P = t_r(b) - t_r(a)$. Without loss of generality, we choose

$$t_r(a) = -P, \quad t_r(b) = 0, \tag{279}$$

which in (278) implies

$$P = \int_a^b \frac{t_\theta - t_r}{r}\mathrm{d}r. \tag{280}$$

Second, the boundary conditions on the two *caps* of the tube, defined as the two rings at $z = 0$ and $z = h$, can be written

$$t_z(z = 0) = N_z, \qquad t_z(z = h) = N_z. \tag{281}$$

However, since a constant axial stretch ζ cannot be used to fit a constant N_z, we replace this point-wise condition on the caps of the cylinder by an integral condition for the total axial load applied on the cap [129, 130]

$$2\pi\int_a^b N_z r dr = N \equiv F + \chi P\pi a^2, \tag{282}$$

thereby eliminating the explicit dependence on the variable r. The total axial load N is further decomposed into an external applied load F, that corresponds to pulling or compressing the tube, and the load created by the internal pressure acting over the cap. That load is simply the pressure times the projected area of the cap. The coefficient χ is 1 for a capped cylinder, and 0 for an infinite cylinder [23, 57].

For incompressible materials, this last expression is not the most practical one as the term T_{zz} will contain an arbitrary pressure. An equivalent expression can be obtained by adding and subtracting T_{rr} to obtain

$$2\pi \int_a^b T_{zz} r \,\mathrm{d}r = 2\pi \int_a^b (T_{zz} - T_{rr} + T_{rr}) r \,\mathrm{d}r. \tag{283}$$

The last term can be integrated by parts, and use of the balance law (339) gives

$$2\pi \int_a^b T_{zz} r \mathrm{d}r = \pi \int_a^b (2T_{zz} - T_{rr} - T_{\theta\theta}) r \mathrm{d}r + P\pi a^2. \tag{284}$$

which implies

$$\pi \int_a^b (2T_{zz} - T_{rr} - T_{\theta\theta}) r \mathrm{d}r = F + (\chi - 1) P\pi a^2, \tag{285}$$

and the last term vanishes for a capped cylinder.

To close the system, we use the constitutive law:

$$t_r = \lambda_r \frac{\partial W}{\partial \lambda_r} - p, \quad t_\theta = \lambda_\theta \frac{\partial W}{\partial \lambda_\theta} - p, \quad t_z = \lambda_z \frac{\partial W}{\partial \lambda_z} - p, \tag{286}$$

and substitute the functions of $(\lambda_r = 1/(\lambda\zeta), \lambda_\theta = \lambda, \lambda_z = \zeta)$ in (280-282) so that $t_\theta - t_r = Q(r, \lambda_a, \zeta)$.

The semi-inverse problem consists then in finding the values of (λ_a, ζ) corresponding to the two external loads (F, P) through the analysis of the two equations (280-282). Note that once (λ_a, ζ) is known, the stress $t_r(r)$ can be computed as

$$t_r(r) = -P + \int_a^r Q(r, \lambda_a, \zeta) \,\mathrm{d}r, \tag{287}$$

and the remaining stresses are obtained from (286).

7.4.1 A toy model for an artery

As an example, we consider a toy model for an artery subject to pressure and tensile stretch. The artery is modeled as a capped tube ($\chi = 1$) made of a Fung material with values taken from Table 1. The system is subjected to an fixed axial force F and varying pressure P [149]. We solve Equations (280)-(282) for (λ_a, ζ) and plot P as a function of ζ, so that if the pressure is controlled, one can determine the amount of axial stretch in the tube.

First, we consider the case of a neo-Hookean material. The curve $P(\zeta)$ shown in Figure 14 is non-monotonic and presents a maximal value of pressure after which unbounded extensional growth follows. This behavior is a drawback of the neo-Hookean model which is not well defined for arbitrarily large deformations and stresses.

Second, we consider the effect of the strain-stiffening parameter β. In Figure 15, we see that for small values of β, a non-monotonic behavior is observed in moderate deformations, followed by a rapidly increasing pressure for larger deformations. For larger values of the strain-stiffening parameter, the behavior is monotonic and increasingly large pressures are needed for further small incremental extensions.

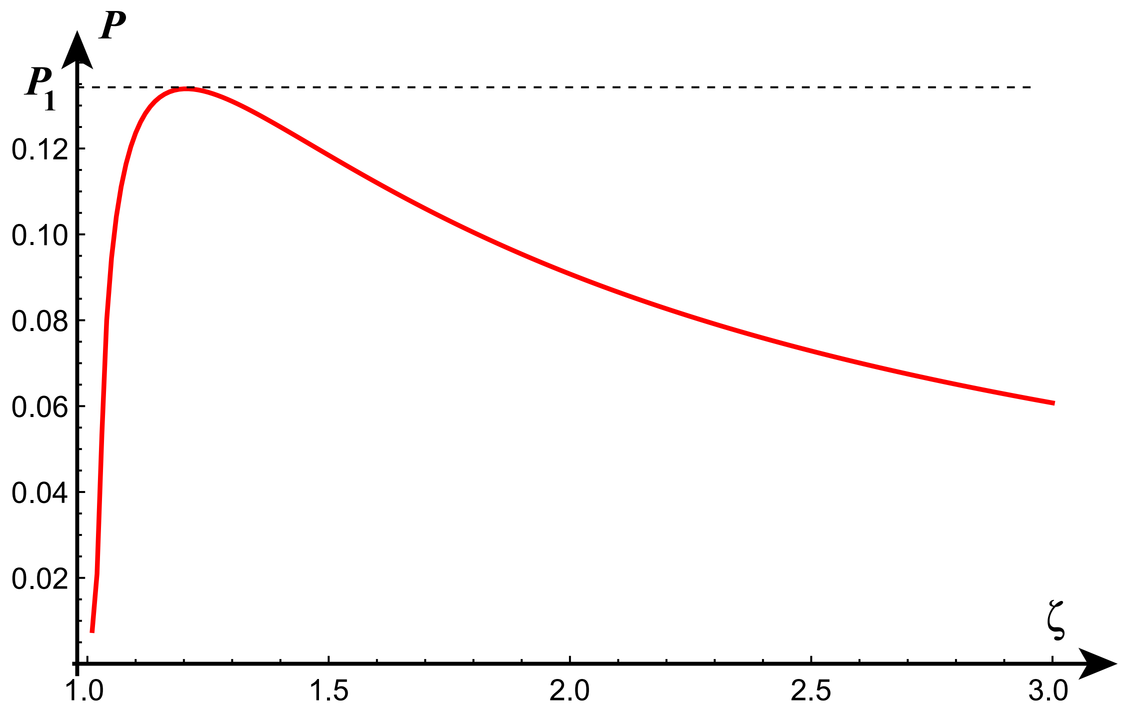


Figure 14: Behavior of a tube under pressure and axial force for a Neo-Hookean material ($A = 1, B = 1.2, \mu = 1, F = 0.1, \chi = 1$). The neo-Hookean model is ill defined past a critical pressure P_1.

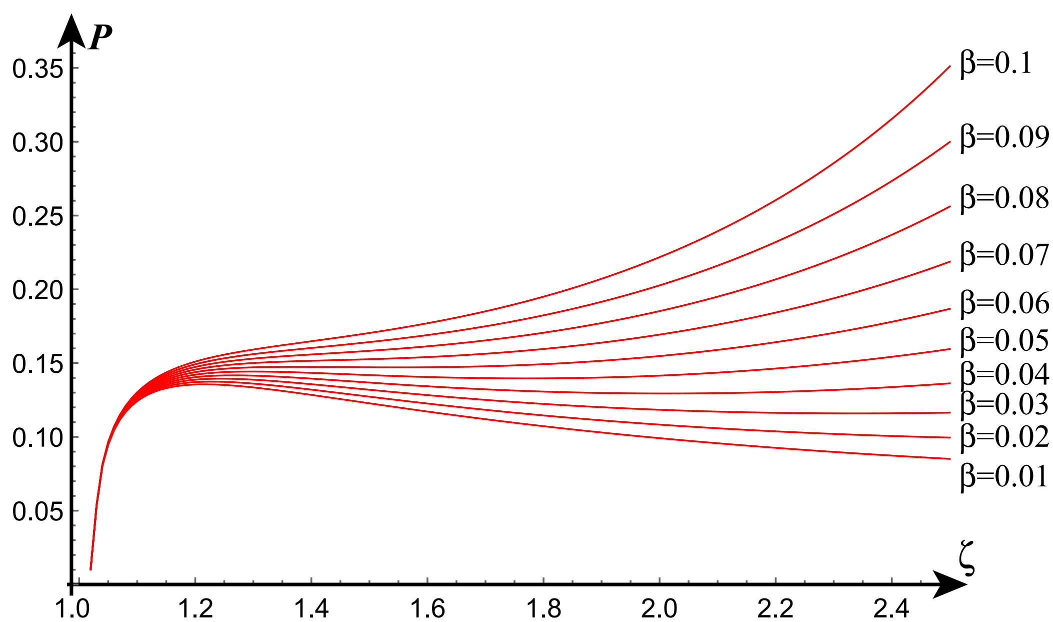


Figure 15: Behavior of a tube under pressure and axial force for a Fung material ($\beta > 0$). Parameters for all figures:$A = 1, B = 1.2, \mu = 1, F = 0.1, \chi = 1$.

We conclude that the behavior of the neo-Hookean model is qualitatively different from the behavior of the Fung model even for very small values of the strain-stiffening parameter β. In particular, typical axial strains in arteries are around 1.3 to 1.6, in the region where the response of the structure clearly depends on the choice of the material model. This simple computation clearly demonstrates the importance of the choice of the material model for a given problem and the need to use nonlinear rather than linear elasticity to study these problems.

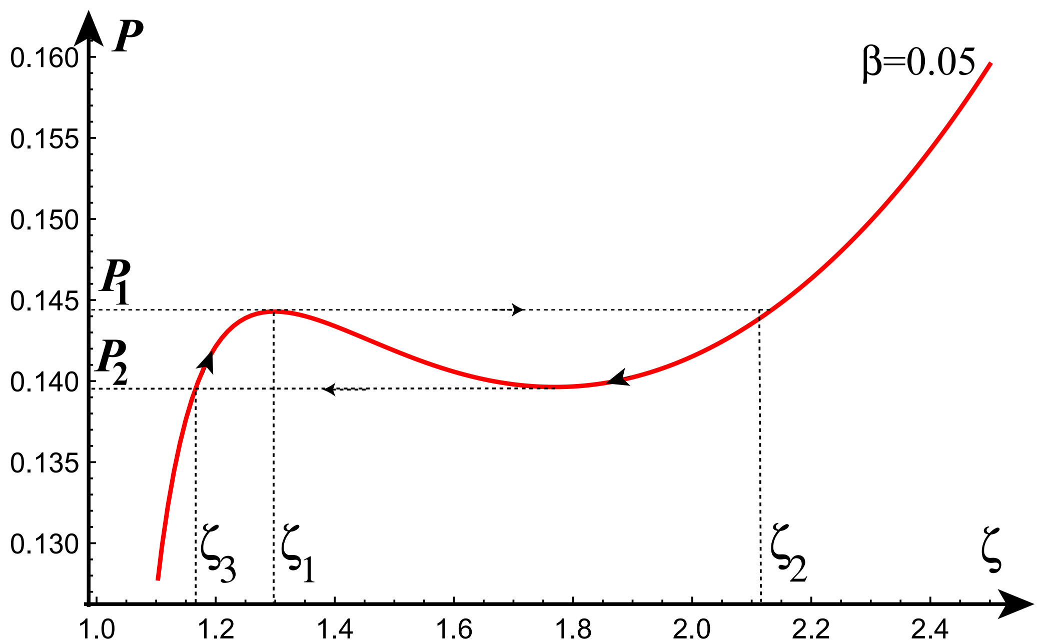


Figure 16: Limit-point instability in of a tube under pressure and axial force for a Fung material ($\beta = 0.05$). The curve $P(\zeta)$ has a minimum and a maximum. If the internal pressure of the tube is raised to P_1, a sudden extension occurs through a jump from ζ_1 to ζ_2. When the pressure is then decreased to P_2, a second jump occurs to ζ_3 (lower arrow).

The non-monotonic behavior shown in Figure 16, for intermediate values of β, is the well-known *limit-point instability* that already appeared in the experiment by Osborne and Sutherland (compare Figure 1 with Figure 16) for spherical and cylindrical shells under internal pressure [2, 6, 22, 54, 116, 117]. For certain materials, the pressure-stretch curve for spherical shells may present a maximum, followed by a minimum; in that case, once the maximum is reached, and the pressure is increased, the stretch will "jump" to a significantly higher value. In spherical shells, several authors have shown that this limit-point instability disappears as the strain-hardening parameter is increased [11, 15, 83, 118].

It is tempting to associate certain pathologies such as aneurysms appearing in arteries to a possible mechanical instability. However, the stabilizing effect due to strain-stiffening leads us to conclude that a simple explanation for the formation of aneurysms in terms of limit-point instability is not plausible [26, 91]. The same general trend is exhibited by more realistic models of arteries, involving multiple layers, fiber anisotropy, and residual stress as we will consider in Section **??**. Nevertheless, various authors have looked at the interesting possibility that aneurysms could be triggered by a *local* mechanical instability resulting in a localized, but stable, bulged configuration that would then evolve slowly and remodel [46, 47, 48, 123, 124, 131]. Aneurysms are such complex progressive diseases that it is unlikely that their formation could be explained simply as a mechanical phenomenon. Still, it is now appreciated that mechanics and mechanical feedback play an important role in the proper function of arteries and in the formation of aneurysms process [52, 127].

8 Universal deformations for isotropic materials

■ **Overview** Only a few problems have exact solution in isotropic incompressible solutions. Here we give a complete list of all known solutions. An open problem is to determine if this list is complete.

One of the advantages of the general formalism of nonlinear elasticity is the possibility of solving boundary-value problems for arbitrary strain-energy functions. However, it is not clear that a given class of deformations can exist for any strain-energy function in the absence of body loads. A *semi-inverse problem* consists in specifying a class of deformations (for instance the inflation of a sphere) with several unknown functions or constants that are determined through the equilibrium equations. The question is then to determine all transformations that can be effected through boundary traction in every homogeneous isotropic hyperelastic material, and in the absence of body forces, the so-called *universal deformations* for which the semi-inverse problem is well-defined [10, 36, 37].

For compressible materials, Ericksen [37] proved that the homogeneous deformations given in Section 7.1 are the only possible universal deformations. That is, without further restricting the class of strain-energy functions or applying body loads, homogeneous deformations are the only ones that can be sustained for arbitrary strain-energy functions [125].

For incompressible materials, a number of interesting universal deformations are known, but the general problem of determining all such possible deformations is still open [10, 41]. In addition to homogenous deformations (classified as Family 0), there are five known families of universal deformations. Each family of solutions exists for all strain-energy functions and suitable boundary tractions. Since the deformation is known, the boundary tractions needed to maintain a given solution can be found by evaluating the Cauchy stress tensor at the boundary. Doing so, one is able to relate the parameters appearing in the solution to the loads required to maintain them.

- **Family 0.** Homogeneous deformations of a rectangular block:

$$x_i = F_{ij}X_j, \qquad i = 1, 2, 3, \tag{288}$$

where (X_1, X_2, X_3) and (x_1, x_2, x_3) are the Cartesian coordinates of a material point in the reference and current configurations, respectively. The deformation is specified by the nine constants appearing in the deformation gradient $\mathbf{F}$.

- **Family 1.** Bending, stretching, and shearing of a rectangular block (shown in Figure 17). This deformation is defined with three arbitrary constants (a, b, c) $(ab \neq 0)$ by

$$r = \sqrt{2aX}, \quad \theta = bY, \quad z = \frac{Z}{ab} - bcY, \tag{289}$$

where (X, Y, Z) and (r, θ, z) are the Cartesian and cylindrical coordinates of a material point in the reference and current configurations, respectively.

In these coordinates, the matrix of components of the deformation gradient reads (for $a > 0$):

$$[\mathbf{F}] = \begin{bmatrix} \sqrt{a}/\sqrt{2X} & 0 & 0 \\ 0 & b\sqrt{2aX} & 0 \\ 0 & -bc & 1/(ab) \end{bmatrix}. \tag{290}$$

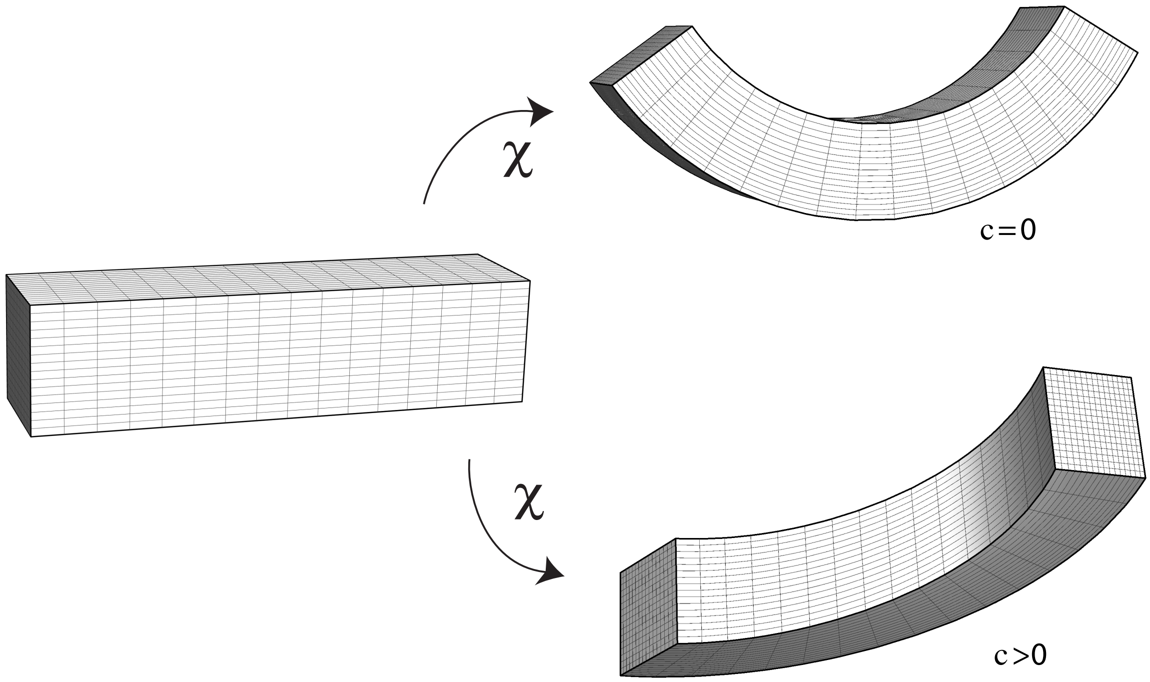


Figure 17: Family 1. Bending, stretching, and shearing of a rectangular block. When the parameter c vanishes, the deformation corresponds to the bending/stretching of a block. The parameter c is associated to shearing out of the bending plane.

- **Family 2.** Straightening, stretching, and shearing of a sector of a cylindrical shell, depicted in Figure 18, and defined for three arbitrary constants (a, b, c) $(ab \neq 0)$ by

$$x = \frac{1}{2}ab^2R^2, \quad y = \frac{\Theta}{ab}, \quad z = \frac{Z}{b} - c\frac{\Theta}{ab}, \tag{291}$$

  where (R, Θ, Z) and (x, y, z) are the cylindrical and Cartesian coordinates of a material point in the reference and current configurations, respectively.

  In these coordinates, the matrix of components of the deformation gradient is given by

$$[\mathbf{F}] = \begin{bmatrix} ab^2R & 0 & 0 \\ 0 & 1/(abR) & 0 \\ 0 & -c/(abR) & 1/b \end{bmatrix}. \tag{292}$$

- **Family 3.** Inflation, bending, torsion, extension, and shearing of a sector of an annular wedge, shown in Figure 19, defined by six arbitrary constants (a, b, c, d, e, f) with the constraint $a(cf - de) = 1$:

$$r = \sqrt{aR^2 + b}, \quad \theta = c\Theta + dZ, \quad z = e\Theta + fZ, \tag{293}$$

  where (R, Θ, Z) and (r, θ, z) are the cylindrical coordinates of a material point in the reference and current configurations, respectively. The shearing is both axial (torsional) and azimuthal. That is the region at constant Z becomes a helicoidal surface.

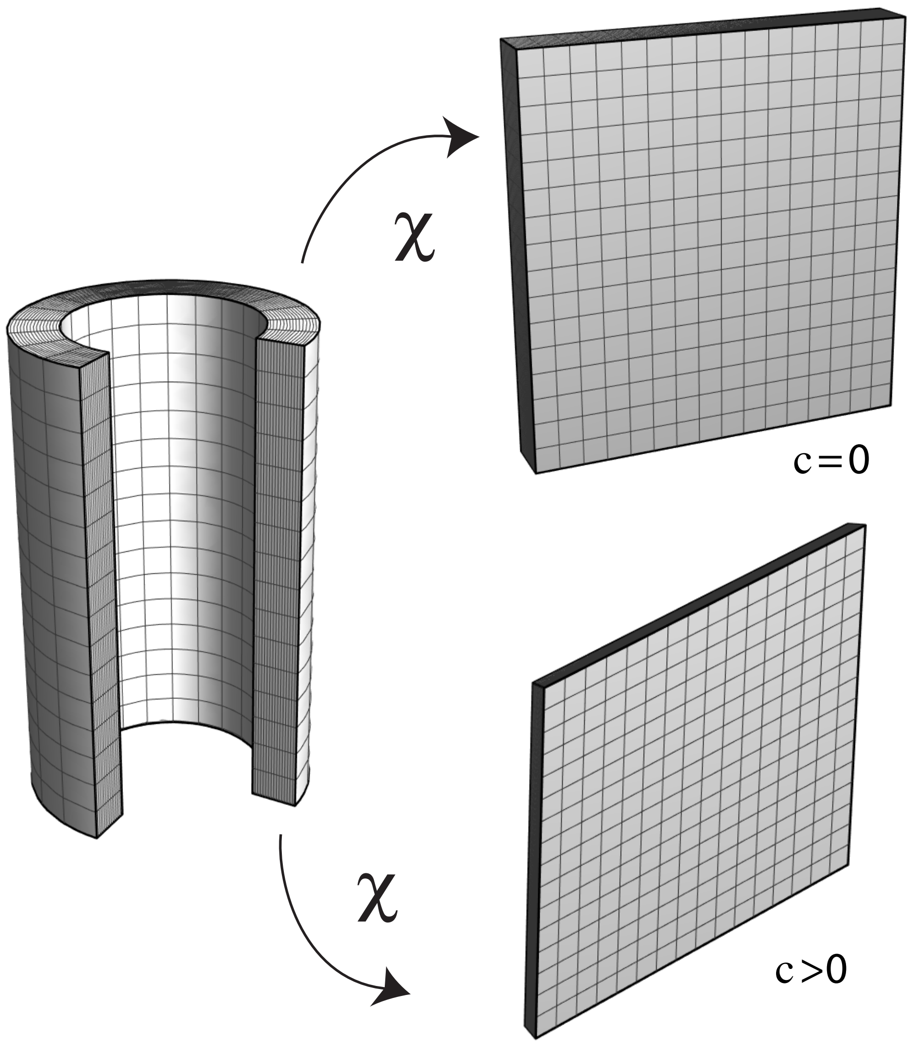


Figure 18: Family 2. Straightening, stretching, and shearing of a sector of a cylindrical shell. When the parameter c vanishes, the deformation corresponds to the straightening of the cylinder. The parameter c is associated to extra shearing of the resulting block.

In these coordinates, the matrix of components of the deformation gradient is given by

$$[\mathbf{F}] = \begin{bmatrix} \frac{aR}{\sqrt{aR^2+b}} & 0 & 0 \\ 0 & \frac{c\sqrt{aR^2+b}}{R} & d\sqrt{aR^2+b} \\ 0 & e/R & f \end{bmatrix}. \tag{294}$$

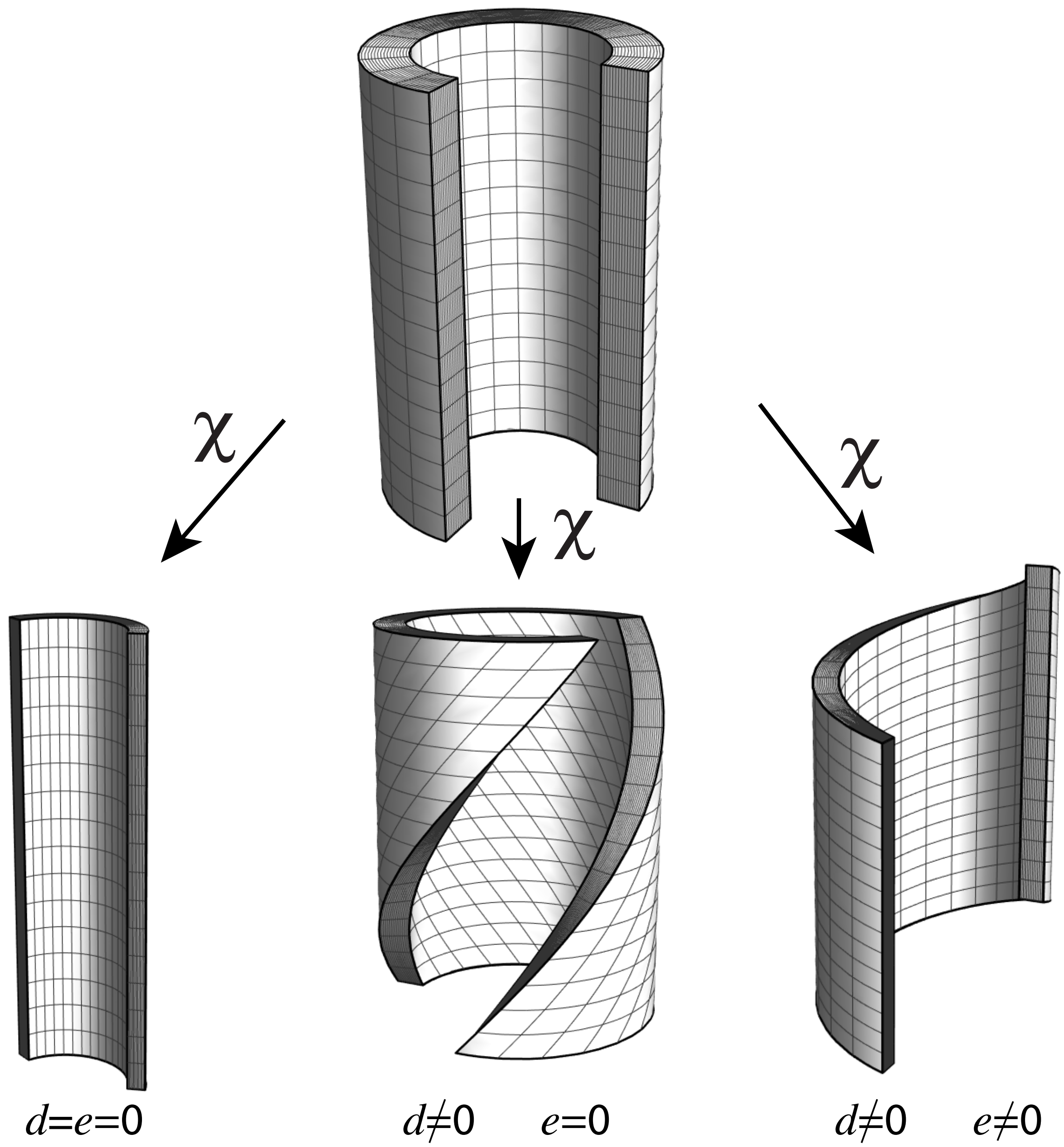


Figure 19: Family 3. Inflation, bending, torsion, extension, and shearing of a cylindrical shell. The parameters d and e control the torsion and the shearing along the axis.

- **Family 4.** Inflation (+) or inversion (-) of a sector of a spherical shell, shown in Figure 20 with a single constant a:

$$r = \sqrt[3]{\pm R^3 + a}, \quad \theta = \pm\Theta, \quad \phi = \Phi, \tag{295}$$

where (R, Θ, Φ) and (r, θ, ϕ) are the spherical coordinates of a material point in the reference and current configurations respectively.

In these coordinates, the matrix of components of the deformation gradient is given by

$$[\mathbf{F}] = \begin{bmatrix} \pm\frac{R^2}{(R^3+a)^{2/3}} & 0 & 0 \\ 0 & \pm\frac{\sqrt[3]{R^3+a}}{R} & 0 \\ 0 & 0 & \pm\frac{\sqrt[3]{R^3+a}}{R} \end{bmatrix}. \tag{296}$$

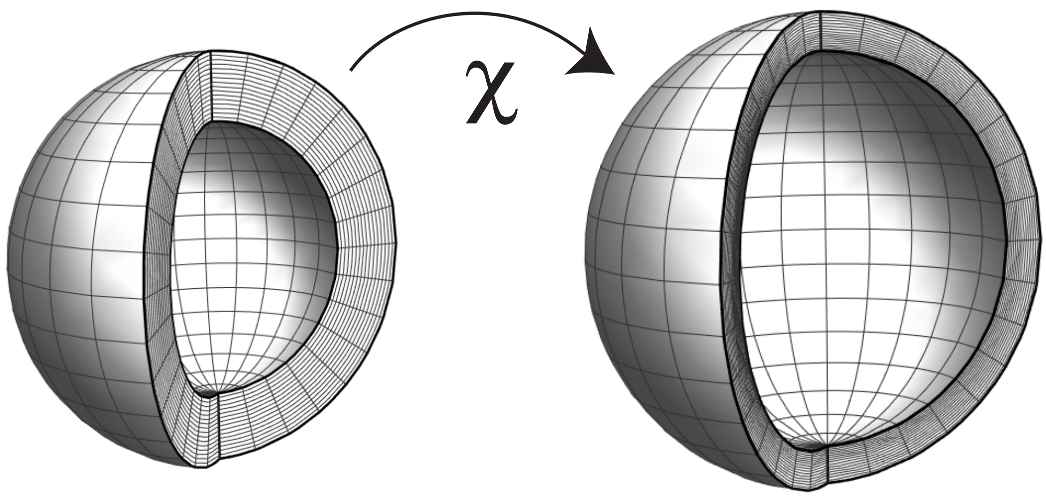


Figure 20: Family 4. Inflation of a spherical shell to another spherical shell. The deformation is controlled by a single parameter. A possible choice is the ratio of current to reference inner radii.

- **Family 5.** Inflation, bending, extension, and azimuthal shearing of an annular wedge, shown in Figure 21, with five constants (a, b, c, d, e) constrained by the condition $a^2ce = 1$:

$$r = aR, \quad \theta = c\Theta + d\log(bR), \quad z = eZ, \tag{297}$$

where (R, Θ, Z) and (r, θ, z) are the cylindrical coordinates of a material point in the reference and current configurations, respectively.

In these coordinates, the matrix of components of the deformation gradient is given by

$$[\mathbf{F}] = \begin{bmatrix} a & 0 & 0 \\ ad & ac & 0 \\ 0 & 0 & e \end{bmatrix}. \tag{298}$$

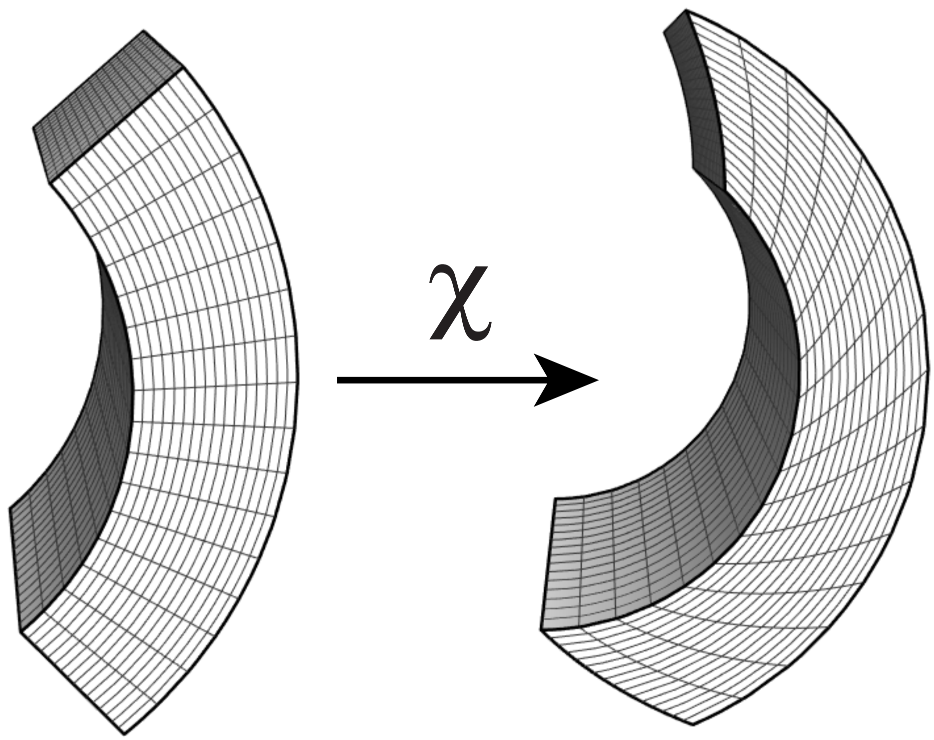


Figure 21: Family 5. Inflation, bending, extension, and azimuthal shearing of an annular wedge. Note that the deformation is essentially planar since the deformation in the z–direction is homogenous.

9 Anisotropic materials

■ **Overview** Anisotropic materials are a little more complicated but they are quite interesting as anisotropy is found in many biological systems.

Many elastic biological tissues have highly anisotropic mechanical properties [51, p. 500]. These anisotropic properties are determined primarily by the presence of fibers [42, 156]. For instance, for animal soft tissues such as tendons, arterial wall, aortic valve, myocardium and pericardium, anisotropy is determined in the first place by the arrangement of the collagen fibers [83, 51, 69, 85, 97] in a mostly isotropic elastin matrix.

Collagen is a fibrous protein, which comprises 25% of the total protein mass in mammals [5, p. 1184] and is the most abundant protein in vertebrates. Collagen is present in tissues in various forms, in particular, in the form of fibrils, which consist of many cross-linked collagen molecules, that are from 50 to 500 nm in diameter and can further organize into fascicles. Special collagen type, geometry, density, and arrangement endows tissues with anisotropic mechanical properties. Collagen fibers in tendons are parallel and aligned in the direction of loading, while in the arterial wall, a significant fiber dispersion around two preferred fiber directions is observed. It also provides the tissue with strongly nonlinear mechanical responses as illustrated by the phenomenon of *stretch locking* in rabbit skin [97] shown in Figure 22.

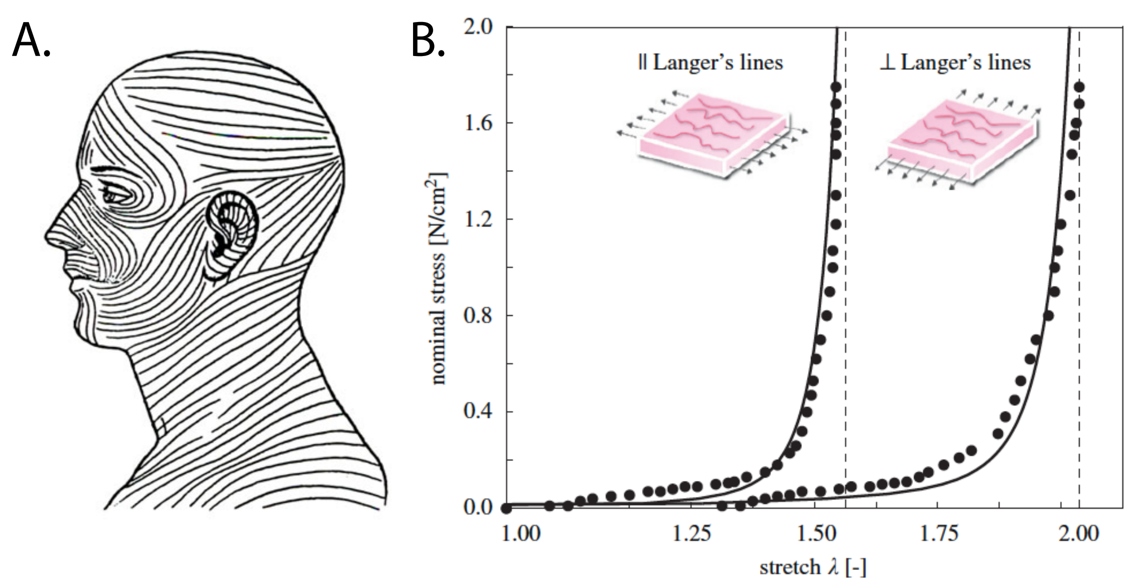


Figure 22: A. Langer's lines showing line of anisotropy in the skin. B. Uniaxial tension test on rabbit skin shows different behaviors in direction along or across Langer's lines (reproduced from [18], based on data [97]). Solid lines represent the computational simulation based on an eight-chain model [8]. The dotted lines are the so-called *locking stretches* after which no further stretch is possible (picture courtesy of Ellen Kuhl).

In plant tissues the same mechanical role is played by cellulose microfibrils based on sugar chains, and with Young's modulus around 130 GPa, reinforcing an isotropic matrix composed of hemicellulose and lignin molecules with respective Young's modulus around 40 MPa and 2 GPa [19]. It is the fine control of fiber geometry and density that provides plants with their mechanical property and their ability to respond to their environment [35, 43, 99].

Similarly, in the fungal kingdom, the cell wall is, typically, constructed of chitin microfibrils embedded in an elastic matrix of amorphous material composed of chitosan and chitin [20, 81]. Such a composite material naturally lends itself to modeling in terms of an elastic matrix with fiber reinforcements [56].

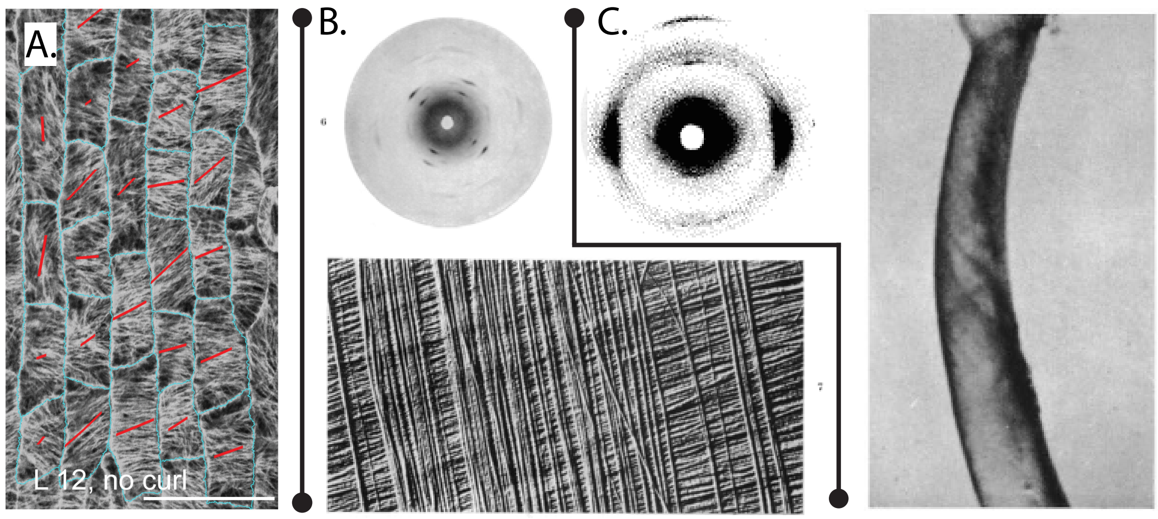


Figure 23: Examples of fiber reinforcement in biological tissues. A. Microtubule alignment in the exocarp cells of *Cardamine hirsuta*. The microtubules wind helically around the cells (picture courtesy of Angela Hay [67]). B. Microfibrils in green algae. The X-ray analysis shows the characteristic X structure indicating a mostly helical fiber [44]. C. Fiber wall structure of the sporangiophore of the fungus Phycomyces showing spiraling fibers in the cytoplasm together with the same characteristic X-ray signature [111].

As the internal structure of a tissue determines its mechanical properties, it is reasonable to include it in a constitutive model. The theory of nonlinear fiber-reinforced elastic composites, developed by Rivlin, Spencer and others [3, 139, 140, 145], asserts that the strain-energy function is in general expressed through a set of deformation invariants, whose number depends on the symmetry exhibited by the material. In this theory, fibers are modeled by continuous fields, that is they are represented by local directions of anisotropy rather than actual physical fibers. A popular, *a priori*, assumption in constitutive models is that the total stress generated by the whole tissue is the sum of stresses generated by its constituents [73, 95, 96]. This assumption can be used to incorporate quantitative data characterizing a tissue's structure directly into the constitutive relation, e.g. fiber volume fraction or orientation-dependent density can be included as multiplicative factors in the appropriate term. In addition, structural approaches allow for a formulation of phenomenological laws for fiber remodeling and the study of the dynamics of mechanically induced fiber reorientation [33, 62, 105, 106].

9.1 One fiber

A material reinforced by fibers that are perfectly aligned in one direction is an example of a *transversely isotropic material*, i.e. a material which has one distinguished direction. The fiber direction is specified by a unit vector $\mathbf{M}$ in the reference configuration as shown in Figure 24. For this fiber, we define the *structure tensor*

$$\mathbf{H} = \mathbf{M} \otimes \mathbf{M}, \tag{299}$$

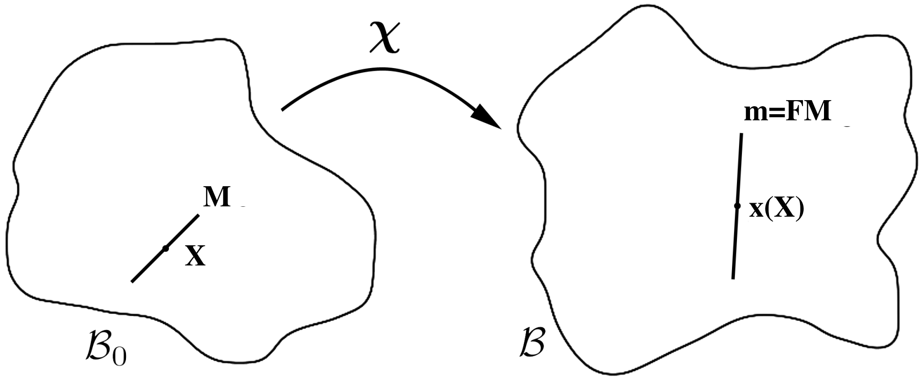


Figure 24: A one-fiber material. In an anisotropic one-fiber material, the response of the material depends on the deformation gradient $\mathbf{F}$ and the direction of a distinguished material line represented by a unit vector $\mathbf{M}$. In a deformation, this direction is mapped to a vector $\mathbf{m} = \mathbf{FM}$.

that contains all information about material anisotropy. We assume that the strain-energy function of the fiber-reinforced material depends on $\mathbf{H}$, and as a consequence, the requirement of isotropy $W(\mathbf{Q}^\mathsf{T}\mathbf{C}\mathbf{Q}) = W(\mathbf{C})$ is not identically satisfied for an arbitrary proper orthogonal second order tensor $\mathbf{Q}$. Instead, it must be satisfied for all proper orthogonal tensors $\mathbf{Q}$ such that $\mathbf{QM} = \pm\mathbf{M}$. This condition is enforced by considering that the strain-energy function is a function of both $\mathbf{C}$ and $\mathbf{H}$, that is $W = W(\mathbf{C}, \mathbf{H})$, and demanding that

$$W(\mathbf{C}, \mathbf{H}) = W(\mathbf{Q}^\mathsf{T}\mathbf{C}\mathbf{Q}, \mathbf{Q}^\mathsf{T}\mathbf{H}\mathbf{Q}), \quad \forall \mathbf{Q} \in SO(3). \tag{300}$$

A strain-energy function for such materials depends in general on a deformation tensor through five scalars, which consists of the three usual isotropic deformation invariants I_1, I_2, I_3 defined by

$$I_1 = \operatorname{tr}\mathbf{B}, \quad I_2 = \frac{1}{2}[I_1^2 - \operatorname{tr}(\mathbf{B}^2)], \quad I_3 = \det\mathbf{B} = J^2. \tag{301}$$

and two extra *pseudo-invariants*, which are related to the strains in the fibers when deformed.

$$I_4 = \mathbf{M}\cdot(\mathbf{CM}) = \mathbf{C} : \mathbf{H}, \qquad I_5 = \mathbf{M}\cdot(\mathbf{C}^2\mathbf{M}) = \mathbf{C}^2 : \mathbf{H}, \tag{302}$$

where we used the double contraction between second-order tensors

$$\mathbf{A} : \mathbf{B} = \operatorname{tr}(\mathbf{AB}) = A_{ij}B_{ji}. \tag{303}$$

The invariant I_4 has a natural interpretation as the square of the fiber stretch in the current configuration, that is the norm of $\mathbf{m} = \mathbf{FM}$.

The general form for the Cauchy stress tensor (178) in terms of W remains valid. Using the identities

$$\mathbf{F}\frac{\partial I_4}{\partial \mathbf{F}} = 2\mathbf{FM}\otimes\mathbf{FM} = 2\mathbf{m}\otimes\mathbf{m}, \tag{304}$$

$$\begin{aligned}\mathbf{F}\frac{\partial I_5}{\partial \mathbf{F}} &= 2\mathbf{F}(\mathbf{M}\otimes\mathbf{FCM} + \mathbf{CM}\otimes\mathbf{FM})\\ &= 2(\mathbf{m}\otimes\mathbf{Bm} + \mathbf{Bm}\otimes\mathbf{m}),\end{aligned} \tag{305}$$

we obtain an explicit expression for the Cauchy stress tensor:

$$\mathbf{T} = J^{-1}\left[-p\mathbf{I} + 2W_1\mathbf{B} + 2W_2(I_1\mathbf{B} - \mathbf{B}^2) + 2I_3W_3 + 2W_4\mathbf{m}\otimes\mathbf{m} + 2W_5(\mathbf{m}\otimes\mathbf{Bm} + \mathbf{Bm}\otimes\mathbf{m})\right], \tag{306}$$

where $W_i = \partial W/\partial I_i$, $i = 1,\dots,5$.

9.2 Two fibers

Many biological tissues are reinforced in two directions and, accordingly, can be modeled by a material with two fibers. The mechanical advantage of a tissue with two fibers appear in extension. With a single fiber, an extension in any direction away from the fiber always produces shear. With two fibers, the shear produced by each fiber can be balanced by the other fiber when they have the same material response and their average direction matches the maximal principal direction. Further, the angle between the two fibers can be tuned to change the overall stiffness of the material.

Reinforcing by an additional family of fibers further reduces the symmetry of the material, but it extends the set of invariants from five to nine. These four extra scalars account for the strains in the second family and the coupling between the two fiber families [140]. We use the unit vectors $\mathbf{M}$ and $\mathbf{M}'$ to define two preferred directions in the reference configuration $\mathcal{B}_0$. The energy for an unconstrained material with two families of fibers is a function of eight invariants. These are the principal invariants I_1, I_2, I_3 together with two pseudo-invariants I_4, I_5 that depend on $\mathbf{M}$, two pseudo-invariants I_6, I_7 that depend on $\mathbf{M}'$ and two coupling terms I_8, I_9 defined by

$$\begin{aligned} I_4 &= \mathbf{M}\cdot(\mathbf{CM}), & I_5 &= \mathbf{M}\cdot(\mathbf{C}^2\mathbf{M}),\\ I_6 &= \mathbf{M}'\cdot(\mathbf{CM}'), & I_7 &= \mathbf{M}'\cdot(\mathbf{C}^2\mathbf{M}'),\\ I_8 &= (\mathbf{M}\cdot\mathbf{M}')\mathbf{M}\cdot(\mathbf{CM}'), & I_9 &= (\mathbf{M}\cdot\mathbf{M}')^2. \end{aligned} \tag{307}$$

Note that the last invariant is not a function of the deformation and will only appear as a constant in the strain-energy function. It will therefore be ignored from the analysis at the expense of a possible re-definition of the strain-energy function. The strain-energy function W is now a function of all first eight invariants so that we write

$$W = W\left(I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8\right). \tag{308}$$

For an incompressible material we have $I_3 = 1$, and the explicit expression for the Cauchy stress tensor is

$$\begin{aligned} \mathbf{T} &= -p\mathbf{I} + 2W_1\mathbf{B} + 2W_2(I_1\mathbf{B} - \mathbf{B}^2)\\ &+ 2W_4\mathbf{m}\otimes\mathbf{m} + 2W_6\mathbf{m}'\otimes\mathbf{m}'\\ &+ 2W_5(\mathbf{m}\otimes\mathbf{Bm} + \mathbf{Bm}\otimes\mathbf{m}) + 2W_7(\mathbf{m}'\otimes\mathbf{Bm}' + \mathbf{Bm}'\otimes\mathbf{m}')\\ &+ W_8(\mathbf{m}\otimes\mathbf{m}' + \mathbf{m}'\otimes\mathbf{m})(\mathbf{M}\cdot\mathbf{M}'), \end{aligned} \tag{309}$$

where $\mathbf{m} = \mathbf{FM}$, $\mathbf{m}' = \mathbf{FM}'$, and $W_i = \partial W/\partial I_i$ for $i = 1,\dots,8$.

9.3 Example: The fiber-reinforced cuboid

We consider the homogeneous deformation of a hyperelastic, incompressible cuboid $\mathcal{B}_0 = [0, L_1]\times[0, L_2]\times[0, L_3]$, subjected to constant normal external loads t_1, t_2, t_3 measured as force per unit area in the deformed configuration. The cuboid is made out of a homogeneous material reinforced by two families of fibers, which are aligned symmetrically in the $\mathbf{E}_1 - \mathbf{E}_2$ plane, as shown in Figure 25. The directions of the fiber families are $\mathbf{M} = (\cos\Theta, \sin\Theta, 0)$, $\mathbf{M}' = (\cos\Theta, -\sin\Theta, 0)$.

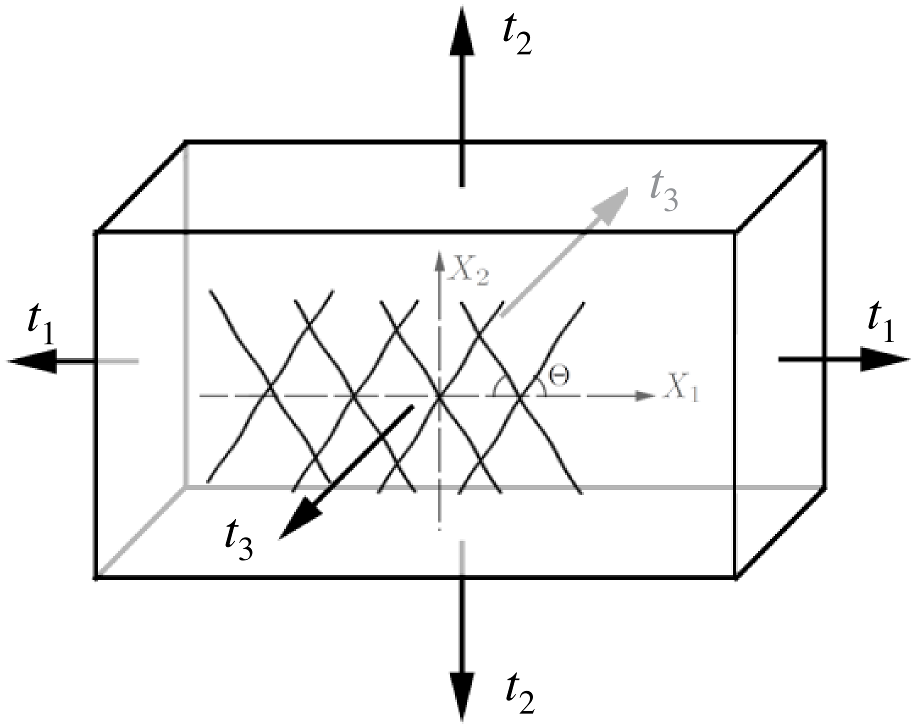


Figure 25: A cuboid is subjected to external hydrostatic loading, maintaining constant normal stress on each face of the cuboid. The material is reinforced by two families of fibers, which are in-plane, aligned symmetrically, and make an angle Θ with the $\mathbf{E}_1$ direction.

We further restrict our attention to a simple form for the isotropic and anisotropic responses, the so-called *standard fiber-reinforcing model* [31, 75, 108, 109, 110, 126, 145]

$$W = W_{\text{iso}} + W_{\text{aniso}} = \frac{\mu}{2}(I_1 - 3) + \frac{\mu\gamma}{2}\left[(I_4 - 1)^2 + (I_6 - 1)^2\right], \tag{310}$$

The constitutive equation (309) for this strain-energy function is

$$\mathbf{T} = -p\mathbf{1} + \mu\mathbf{F}\left[\mathbf{1} + 2\gamma(I_4 - 1)\mathbf{M}\otimes\mathbf{M} + 2\gamma(I_6 - 1)\mathbf{M}'\otimes\mathbf{M}'\right]\mathbf{F}^{\mathsf{T}}. \tag{311}$$

Due to the particular choice of fiber alignment and strength, the two equal and opposite fibers with angle $\pm\Theta$ in the $\mathbf{E}_1$–$\mathbf{E}_2$ plane are mapped into two equal and opposite fibers with angle $\pm\theta$ as shown in Figure 26. We also have $I_4 = I_6 = \lambda_1^2\cos^2\Theta + \lambda_2^2\sin^2\Theta$, where λ_i is the stretch in X_i direction, and

$$[\mathbf{M}\otimes\mathbf{M} + \mathbf{M}'\otimes\mathbf{M}'] = 2\text{diag}(\cos^2\Theta, \sin^2\Theta, 0). \tag{312}$$

Hence, we have

$$\mathbf{T} = -p\mathbf{1} + \mu\mathbf{B}\left[\mathbf{1} + 4\gamma(\lambda_1^2\cos^2\Theta + \lambda_2^2\sin^2\Theta - 1)\text{diag}(\cos^2\Theta, \sin^2\Theta, 0)\right].$$

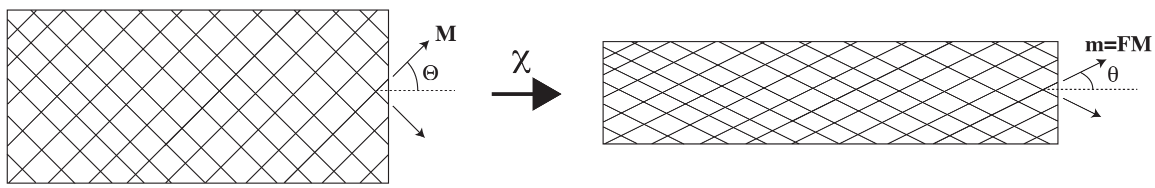


Figure 26: Deformation of two equal and opposite fibers

.

For a homogenous deformation, the Cauchy stress is homogeneous as well and the Cauchy equation are identically satisfied. The particular form of the fiber stiffness and alignment and the boundary conditions

$$\mathbf{T}(X_i = 0) = \mathbf{T}(X_i = L_i) = t_i \mathbf{e}_i, \tag{313}$$

imply that $[\mathbf{T}] = \text{diag}(t_1, t_2, t_3)$. From (9.3), we see that $\mathbf{F}$, in these coordinates, must also be diagonal, that is $[\mathbf{F}] = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$. Thus, (311) together with the incompressibility condition become

$$t_1 = -p + \mu \left(\lambda_1^2 + 4\gamma(\lambda_1^2 \cos^2\Theta + \lambda_2^2 \sin^2\Theta - 1)\lambda_1^2 \cos^2\Theta\right), \tag{314}$$

$$t_2 = -p + \mu \left(\lambda_2^2 + 4\gamma(\lambda_1^2 \cos^2\Theta + \lambda_2^2 \sin^2\Theta - 1)\lambda_2^2 \sin^2\Theta\right), \tag{315}$$

$$t_3 = -p + \mu\lambda_3^2, \tag{316}$$

$$\lambda_1\lambda_2\lambda_3 = 1, \tag{317}$$

where the four unknowns λ_i and p can be determined from the loads t_i and the fiber orientation angle Θ. The last two relations (317) and (316) give $\lambda_3 = \lambda_2^{-1}\lambda_1^{-1}$ and $p = \mu\lambda_3^2 - t_3$, so that, after substitution in the first two equations, the problem can be reduced to finding (λ_1, λ_2) as a function of t_1, t_2, t_3, that is

$$A = \frac{\lambda_1^2 - \lambda_2^2}{4\gamma} + (\lambda_1^2 \cos^2\Theta + \lambda_2^2 \sin^2\Theta - 1)(\lambda_1^2 \cos^2\Theta - \lambda_2^2 \sin^2\Theta), \tag{318}$$

$$B = \frac{-1}{2\gamma\lambda_1^2\lambda_2^2} + \frac{\lambda_1^2 + \lambda_2^2}{4\gamma} + (\lambda_1^2 \cos^2\Theta + \lambda_2^2 \sin^2\Theta - 1)(\lambda_1^2 \cos^2\Theta + \lambda_2^2 \sin^2\Theta), \tag{319}$$

where

$$A = \frac{t_1 - t_2}{4\gamma\mu}, \quad B = \frac{t_1 + t_2 - 2t_3}{4\gamma\mu}. \tag{320}$$

Geometrically, Equations (318-319) define, for given A and B, two level sets, and the solution lies at their intersection.

In many situations, fibers embedded in a matrix may not support compressive loads as they would buckle under compression [73, 141]. Therefore, we further restrict the study of our model to the *fiber-tensile region*, defined by

$$\lambda_f = \lambda_1^2 \cos^2\Theta + \lambda_2^2 \sin^2\Theta > 1, \qquad \Theta \in [0, \pi/2]. \tag{321}$$

Fibers are unstrained on the boundary of the region (321), and are in compression when $\lambda_1^2 \cos^2\Theta + \lambda_2^2 \sin^2\Theta < 1$.

We first compute the effect of fiber alignment and stiffness by computing an effective Young's modulus. Young's modulus is only defined for isotropic material, but in a given direction, one can define a modulus as the ratio of the uniaxial tension N by the stretch in the same direction in the limit of small deformations. That is, the *effective Young's modulus* in a direction is defined as the gradient of N in that direction evaluated at the stress-free state. For example, for an uniaxial tension in the $\mathbf{e}_1$ direction $t_1 = N, t_2 = t_3 = 0$, we define

$$E_{\text{eff}} \equiv \left.\frac{\partial N}{\partial \lambda_1}\right|_{\lambda_1=1}, \tag{322}$$

and, after linearizing (318-319) around the unstressed configuration, we find

$$E_{\text{eff}} = \mu \frac{4\left[3 + 5\gamma + 3\gamma\cos(4\Theta)\right]}{4 + 3\gamma - 4\gamma\cos(2\Theta) + \gamma\cos(4\Theta)}. \tag{323}$$

As shown in Figure 27, the effective Young's modulus has a maximum for $\Theta = 0$ when the

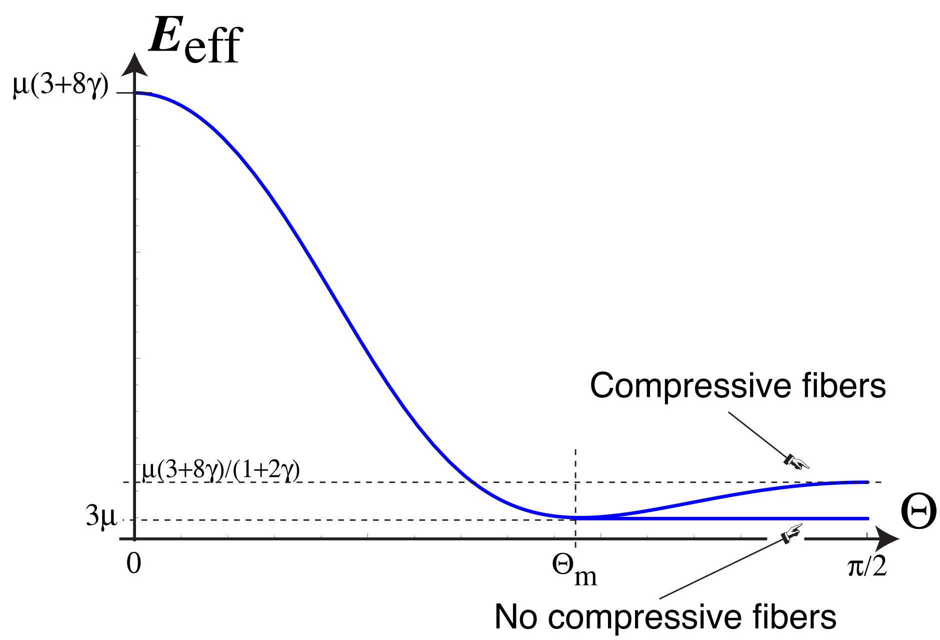


Figure 27: Effective Young's modulus for a fiber-reinforced sheet as a function of the fiber angle.

fibers are aligned with the axis. It then takes the value

$$E_{\text{eff}}(\Theta = 0) = (3 + 8\gamma)\mu. \tag{324}$$

The minimum effective Young's modulus $E_{\text{eff}}(\Theta = \Theta_m) = 3\mu$ is attained for

$$\Theta_{\text{m}} = \tan^{-1}\sqrt{2} = \pi/2 - \Phi_{\text{m}} \approx 54.74°, \tag{325}$$

where $\Phi_{\text{m}} \approx 35.26°$ is the *magic angle* that appears in several interesting applications. It will be discussed in detail in Section 9.5.

For $\Theta_m < \Theta < \pi/2$, $I_4 < 1$ and the fibers are in compression. If we allow for compressive fibers, E_{eff} increases to the locally maximal value of

$$E_{\text{eff}}(\Theta = \pi/2) = \frac{\mu(3 + 8\gamma)}{1 + 2\gamma}. \tag{326}$$

In the absence of compressive fibers, the material behaves isotropically and the effective modulus remains at 3μ.

Remarkably, we see that the apparent stiffness of a material in a given direction can be tuned to any value between the matrix stiffness and the fiber stiffness by choosing the appropriate fiber angle Θ and fiber density. The modulus ratio between the two extremal values is $1 + 3\gamma/8$ and atypical ratio of γ is between 10 and 1000. This simple effect is at work in fungi, plants, and animal tissues where, typically, the isotropic elastic matrix remains mostly unchanged while fibers are constantly turned over to maintain the appropriate level of homeostatic stress, the tissue stiffness or allowing for healing [33, 62, 83, 107].

As an example in the aorta, an elastin matrix reinforced with two equal and opposite fibers helically wrapping around a cylindrical geometry, the effective Young's modulus is around 90 kPa at the ascending aorta and 10 kPa at the femoral bifurcation, which correlates with the elastin content [17].

In the nonlinear regime, we observe an interesting behavior. As expected, the stretch along the tensile stress always increases as shown in Figure 28. Similarly, the stretch perpendicular to the direction of the applied load in the fiber plane always decreases as we would expect for an isotropic material. However, in the direction normal to the fibers, the material thickness determined by λ_3 can increase, decrease, or first decrease then increase. This non-monotonic behavior depends on the balance between the nonlinear material response and the anisotropy.

We can identify in the parameter space, $(1/\gamma, \Theta)$, these different behaviors by solving Equations (318-319) under the condition $\lambda_3 = 1$ that is $\lambda_1 = \lambda_2$. This extra condition leads to two curves in the plane $(1/\gamma, \Theta)$ given by [104]

$$\frac{1}{\gamma} = \sin^2(2\Theta), \qquad \frac{1}{\gamma} = \sin^2(2\Theta) - 4\sin^4\Theta. \tag{327}$$

These curves are shown in Figure 29 together with the corresponding points of Figure 28.

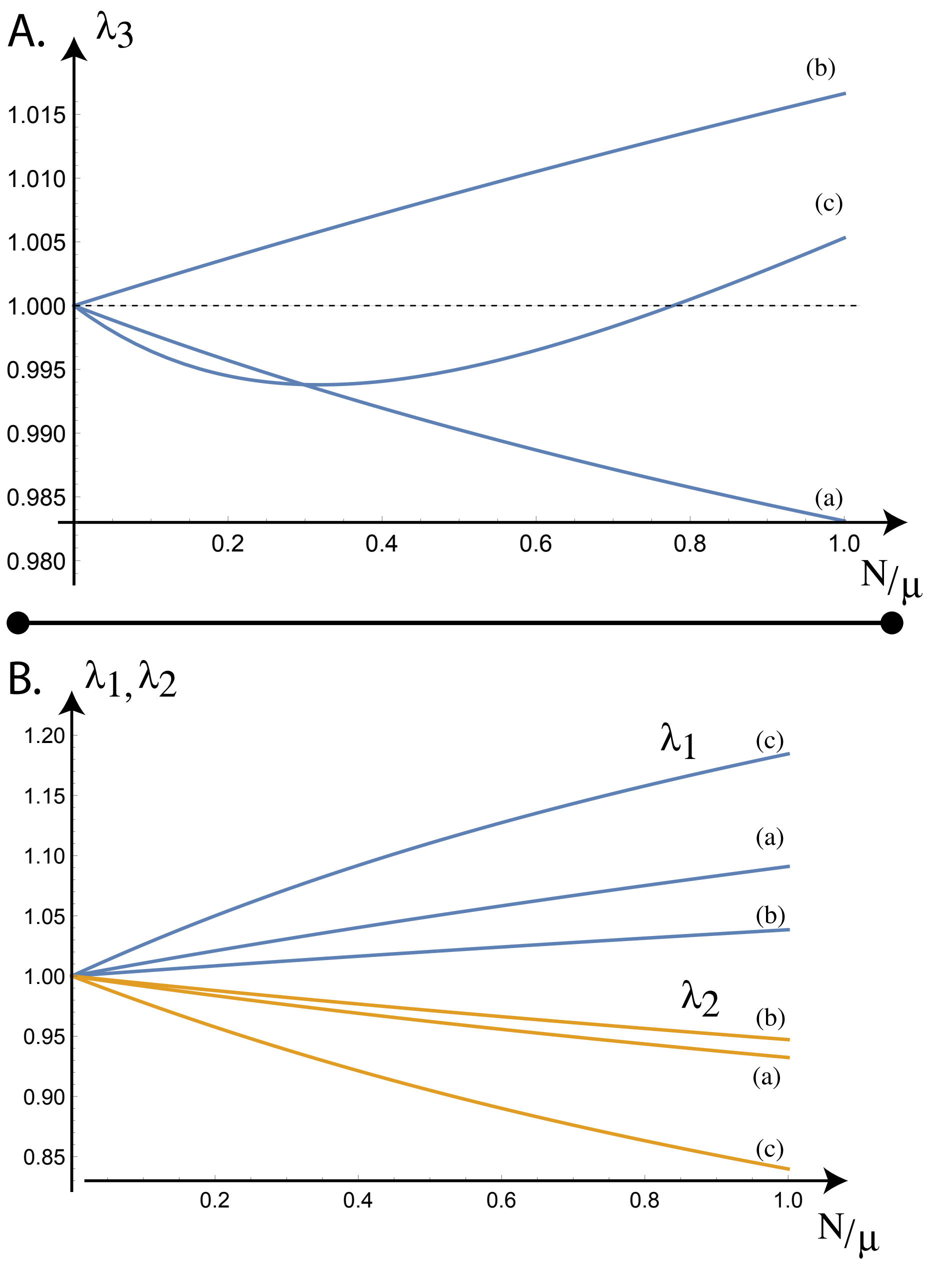


Figure 28: The three principal stretches as a function of the tension. For some values of γ and Θ, the thickness of the cuboid always decreases, always increases, or first decreases, then increases. Parameter values: (a): $\Theta = \pi/8,\ \gamma = 4/3$; (b): $\Theta = \pi/8,\ \gamma = 4$; (c): $\Theta = \pi/4,\ \gamma = 4$.

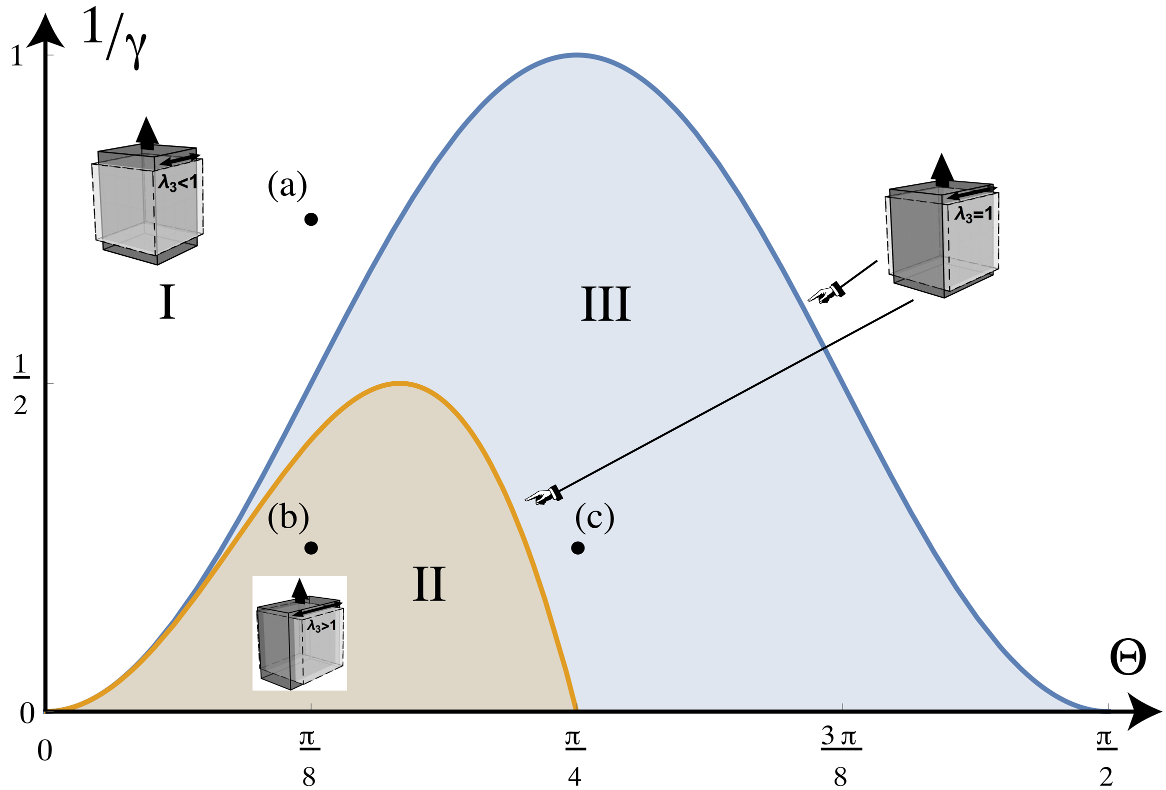


Figure 29: Parameter space for inversion: depending on the relative fiber stiffness γ and the fiber direction Θ, in tension, the thickness of a either decreases (Zone I), increases (Zone II), or first decreases, then increases (Zone III). Insert shown for different values of the tension. The stretch profiles as a function of the tension for specific points (a,b,c) are given in Figure 28.

9.4 Example: The fiber-reinforced cylinder

We now consider a tube made out of an incompressible material with initial inner radius $A = 1$, outer radius $B > A$, and height H deformed into a tube with radii a and b and height h. We assume that the tube is allowed to inflate, extend, and twist while remaining cylindrical at all time. This problem is the classic inflation-extension-torsion problem for the cylinder. It is a particular case of Family 3 of semi-inverse problems (293). In the usual cylindrical coordinates $\{r, \theta, z\}$ and $\{R, \Theta, Z\}$ the deformation is given by

$$r = \sqrt{a^2 + \frac{R^2 - A^2}{\zeta}}, \tag{328}$$

$$\theta = \Theta + \tau\zeta Z, \tag{329}$$

$$z = \zeta Z, \tag{330}$$

where the axial stretch ζ and the twist τ are constant.

The position vectors are, respectively,

$$\mathbf{X} = R\mathbf{E}_R + Z\mathbf{E}_Z, \tag{331}$$

$$\mathbf{x} = \lambda R\mathbf{e}_r + \zeta Z\mathbf{e}_z. \tag{332}$$

Using identity (76), the deformation gradient, $\mathbf{F} = \text{Grad}\,\mathbf{x}$, in cylindrical coordinates is given by

$$[\mathbf{F}] = \begin{bmatrix} \frac{1}{\lambda\zeta} & 0 & 0 \\ 0 & \lambda & \zeta\tau r \\ 0 & 0 & \zeta \end{bmatrix}, \tag{333}$$

where we have used the incompressibility condition $\det \mathbf{F} = 1$ and

$$\lambda = \frac{r}{R} = \frac{1}{R}\sqrt{a^2 + \frac{R^2 - A^2}{\zeta}}. \tag{334}$$

Therefore, a single parameter fully describes the radial profile of the deformation. Setting $\lambda_a = a/A$, it follows that

$$\lambda_b = \frac{b}{B} = \frac{1}{\zeta}\sqrt{1 + \frac{A^2}{B^2}\,(\zeta\lambda_a^2 - 1)}. \tag{335}$$

The anisotropic response of the cylinder is modeled by two families of embedded fibers $\mathbf{M}$ and $\mathbf{M}'$. For simplicity, we will refer to a family of distributed fibers simply as *a fiber*. Both fibers wind helically around the axis and may induce a rotation of the cylinder under extension depending on their strengths and angle. The components of the direction vectors with respect to the basis $(\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z)$ are

$$\begin{bmatrix} M_R \\ M_\Theta \\ M_Z \end{bmatrix} = \begin{bmatrix} 0 \\ \cos\Phi \\ \sin\Phi \end{bmatrix}, \qquad \begin{bmatrix} M'_R \\ M'_\Theta \\ M'_Z \end{bmatrix} = \begin{bmatrix} 0 \\ -\cos\Psi \\ \sin\Psi \end{bmatrix}. \tag{336}$$

Here we have assumed that the fibers remain locally tangent to the cylinder. Following Figure 30, the angles between the fibers and the circumferential direction are denoted by Φ and Ψ. Note that we have chosen the angle Ψ so that when the angles are equal $\Phi = \Psi$, the two fibers make the same angle with the axis, and are said to be *opposite*.

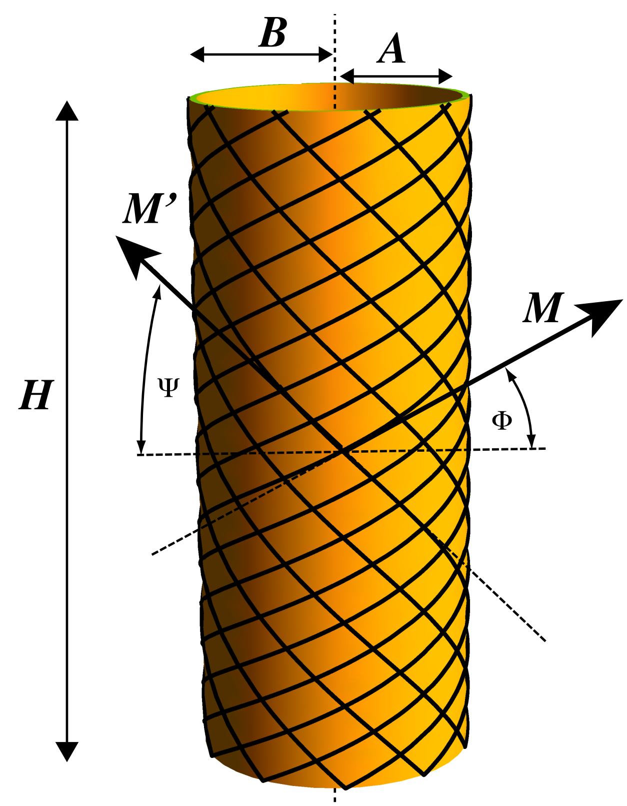


Figure 30: Geometry of the fibers. The angle Φ denote the direction of the first fiber with respect to the cross section (counted counter-clockwise) and the angle Ψ is the angle of the second fiber (counted clockwise).

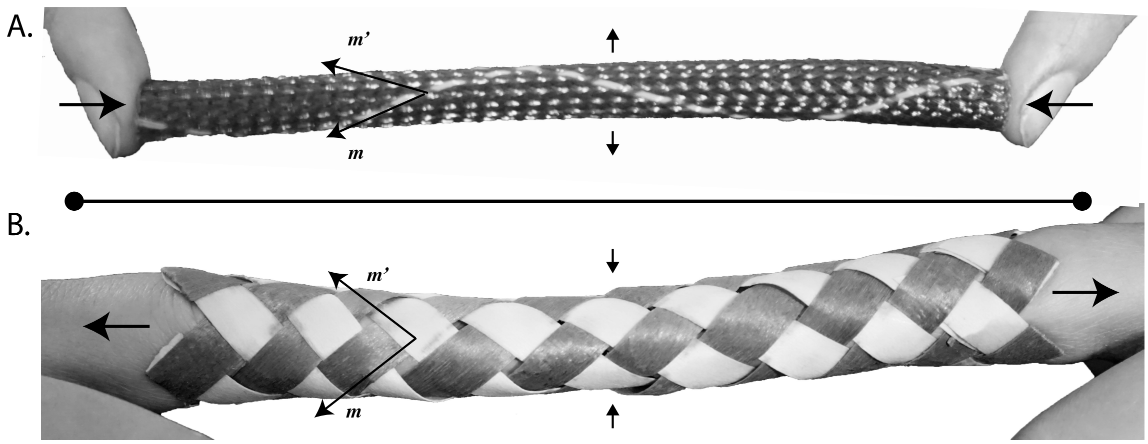


Figure 31: Two simple toys demonstrate the principle of opposite fibers and the change in fiber orientation in compression and tension. Top: The boing-boing rocket stores elastic energy under compression by changing the fiber angle. The rocket can be released by quickly removing one finger. Bottom: the finger trap. In extension, the cylinder radius decreases and traps the fingers.

Opposite fibers are a common occurrence and they are found, for instance, in the popular toys shown in Figure 31. Under a deformation $\mathbf{F}$, the orientation of the fiber characterized by a vector $\mathbf{M}$ with angle Φ in the reference configuration is mapped, in the current configuration, to the vector

$$\mathbf{m} = \begin{bmatrix} m_r \\ m_\theta \\ m_z \end{bmatrix} = \mathbf{FM} = \begin{bmatrix} 0 \\ \lambda \cos \Phi + r\zeta\tau \sin \Phi \\ \zeta \sin \Phi \end{bmatrix}. \tag{337}$$

Therefore, the new fiber angle is

$$\phi = \arctan \left(\frac{\zeta \sin \Phi}{\lambda \cos \Phi + r\zeta\tau \sin \Phi} \right). \tag{338}$$

For these deformations, the only non-vanishing component of the Cauchy equation $\operatorname{div} \mathbf{T} = 0$ is

$$\frac{\mathrm{d}T_{rr}}{\mathrm{d}r} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) = 0. \tag{339}$$

This equation can be integrated once over r

$$T_{rr}(r) = \int_r^b \frac{T_{rr} - T_{\theta\theta}}{r} \, \mathrm{d}r, \qquad a \leqslant r \leqslant b. \tag{340}$$

We consider a simple thought experiment in which the tube is capped at both ends and subject to an axial extension ζ due to an internal pressure P and to a total axial load N on the top cap. The tube is also subject to an external moment M leading to a torsion represented by τ.

First, we express the boundary condition in the radial direction. Taking the radial component of the Cauchy stress tensor $\mathbf{T}$ to vanish at the outer boundary, we have $T_{rr}(r = b) = 0$. On the inner wall, the boundary condition associated with the pressure is $T_{rr}(r = a) = -P$.

Therefore, the two conditions on the pressure can be used to simplify (340) to

$$P = \int_a^b \frac{T_{\theta\theta} - T_{rr}}{r}\,\mathrm{d}r. \tag{341}$$

Second, the condition on the two end caps is a combination of an external axial stress superimposed on the pointwise stress due to the internal pressure acting on the end cap:

$$T_{zz}(z=0) = N_z, \qquad T_{zz}(z=h) = N_z, \tag{342}$$

Following the discussion of Section 7.4, we replace these point-wise conditions by an integral condition relating the total application of forces and moments on the caps of the cylinder:

$$2\pi \int_a^b T_{zz} r\,\mathrm{d}r = N = F + \chi P \pi a^2, \tag{343}$$

where the total axial load N is decomposed into an external applied load F and the load created by the internal pressure acting over the cap. Following the argument in Section 7.4, this last condition can be replaced by

$$\pi \int_a^b (2t_z - t_r - t_\theta) r dr = F + (\chi - 1) P \pi a^2, \tag{344}$$

and the last term vanishes for a capped cylinder as case considered here.

Third, when $\tau \neq 0$, we have to take into account the possibility of applying a moment on the ends. This loading can be expressed also as integral condition relating the total moment acting on the tube axis to the axial stress. That is,

$$\int_a^b T_{\theta z} r^2\,\mathrm{d}r = M. \tag{345}$$

Therefore, the three boundary conditions are:

$$C_1: \quad \int_a^b \frac{T_{\theta\theta} - T_{rr}}{r}\,\mathrm{d}r = P, \tag{346}$$

$$C_2: \quad \pi \int_a^b (2T_{zz} - T_{rr} - T_{\theta\theta}) r\,\mathrm{d}r = F, \tag{347}$$

$$C_3: \quad 2\pi \int_a^b T_{\theta z} r^2\,\mathrm{d}r = M. \tag{348}$$

The semi-inverse problem consists in finding the values of (λ_a, ζ, τ) corresponding to the three external loads (F, M, P) through the analysis of equilibria.

9.4.0.1 The standard fiber-reinforcing model with fiber extension. We further restrict our attention to the *standard fiber-reinforcing model*

$$W_{\text{iso}} = \frac{\mu_1}{2}(I_1 - 3), \tag{349}$$

$$W_{\text{fib}}(I_4) = \frac{\mu_4}{4}(I_4 - 1)^2, \quad W_{\text{fib}}(I_6) = \frac{\mu_6}{4}(I_6 - 1)^2, \tag{350}$$

where the material parameters $\mu_i > 0$ have the dimension of a pressure.

From the strain-energy function, we compute the Cauchy stress tensor

$$\mathbf{T} = \mathbf{F}\frac{\partial W}{\partial \mathbf{F}} - p\mathbf{1}, \tag{351}$$

where p is the Lagrangian multiplier associated with incompressibility. In our case, these equations simplify to

$$\mathbf{T} = 2W_1\mathbf{B} + 2W_4\mathbf{m}\otimes\mathbf{m} + 2W_6\mathbf{m}'\otimes\mathbf{m}' - p\mathbf{1}, \tag{352}$$

with $W_i = \partial_{I_i}W$. The non-vanishing components of the Cauchy stress tensor are given by

$$\begin{aligned}
T_{rr} &= -p + 2W_1\zeta^{-2}\lambda^{-2},\\
T_{\theta\theta} &= -p + 2\left(\lambda^2 + r^2\zeta^2\tau^2\right)W_1\\
&\qquad + 2(\lambda\cos\Phi + r\zeta\tau\sin\Phi)^2W_4 - 2(\lambda\cos\Psi - r\zeta\tau\sin\Psi)^2W_6,\\
T_{zz} &= -p + 2\zeta^2W_1 + 2\zeta^2\sin\Phi^2W_4 + 2\zeta^2\sin\Psi^2W_6,\\
T_{z\theta} &= T_{\theta z} = 2\zeta\left[r\zeta\tau W_1 + \sin\Phi(\lambda\cos\Phi + r\zeta\tau\sin\Phi)W_4\right.\\
&\qquad\qquad \left. - \sin\Psi(\lambda\cos\Psi - r\zeta\tau\sin\Psi)W_6\right].
\end{aligned}$$

Since the constitutive relationships are written in terms of $\{\lambda,\zeta,\tau\}$, we rewrite the three boundary conditions in terms of integrals over λ by using the identity

$$\frac{\mathrm{d}r}{\mathrm{d}\lambda} = A\frac{(1-\zeta\lambda_a^2)^{1/2}}{(1-\zeta\lambda^2)^{3/2}}, \tag{353}$$

which yields the equivalent boundary conditions

$$C_1: \quad \int_{\lambda_a}^{\lambda_b} \frac{T_{rr}-T_{\theta\theta}}{\lambda(\lambda^2\zeta-1)}\,\mathrm{d}\lambda = P, \tag{354}$$

$$C_2: \quad \pi A^2\int_{\lambda_a}^{\lambda_b} \frac{1-\zeta\lambda_a^2}{(1-\zeta\lambda^2)^2}\lambda(2T_{zz}-T_{rr}-T_{\theta\theta})\,\mathrm{d}\lambda = F, \tag{355}$$

$$C_3: \quad 2\pi A^3\int_{\lambda_a}^{\lambda_b} \frac{(1-\zeta\lambda_a^2)^{3/2}}{(1-\zeta\lambda^2)^{5/2}}\lambda^2T_{z\theta}\,\mathrm{d}\lambda = M. \tag{356}$$

While explicit expression for the three integrals for (M,N,P) for the particular choice (349-350) can be obtained, they are far too cumbersome to be useful.

9.4.0.2 Membrane limit. We can take advantage of the assumption that the tube is thin and expand the three integrals (M,N,P) in the thickness of the tube. Without loss of generality, we measure all lengths with respect to the inner reference radius, that is we set $A=1$. Then, we introduce ϵ by $B=1+\epsilon$ and expand

$$M = M^{(1)}\epsilon + M^{(2)}\epsilon^2 + \ldots, \tag{357}$$

$$F = F^{(1)}\epsilon + F^{(2)}\epsilon^2 + \ldots, \tag{358}$$

$$P = P^{(1)}\epsilon + P^{(2)}\epsilon^2 + \ldots. \tag{359}$$

Explicitly, to first order these expressions read

$$M^{(1)} = 2\pi\lambda\left[\zeta\lambda\mu_1\tau + \mu_4 J_4\sin(\Phi)(\zeta\lambda\tau\sin(\Phi) + \lambda\cos(\Phi)) + \mu_6 J_6\sin(\Psi)(\zeta\lambda\tau\sin(\Psi) - \lambda\cos(\Psi))\right], \tag{360}$$

$$F^{(1)} = -\frac{\pi}{\zeta}\left[\frac{\mu_1}{\zeta^2\lambda^2}\left(1 + \zeta^4\lambda^2\left(\lambda^2\tau^2 - 2\right) + \zeta^2\lambda^4\right) + \mu_4 J_4\left(\zeta\sin(\Phi)\left(\zeta\left(\lambda^2\tau^2 - 2\right)\sin(\Phi) + 2\lambda^2\tau\cos(\Phi)\right) + \lambda^2\cos^2(\Phi)\right) + \mu_6 J_6\left(\zeta^2\left(\lambda^2\tau^2 - 2\right)\sin^2(\Psi) - 2\zeta\lambda^2\tau\sin(\Psi)\cos(\Psi) + \lambda^2\cos^2(\Psi)\right)\right], \tag{361}$$

$$P^{(1)} = \frac{1}{\zeta}\left[\frac{\mu_1}{\zeta^2\lambda^4}\left(\zeta^4\lambda^4\tau^2 + \zeta^2\lambda^4 - 1\right) + \mu_4 J_4(\zeta\tau\sin(\Phi) + \cos(\Phi))^2 + \mu_6 J_6(\cos(\Psi) - \zeta\tau\sin(\Psi))^2\right], \tag{362}$$

where

$$\begin{aligned} J_4 &= (I_4 - 1) \\ &= \zeta\sin(\Phi)\left(\zeta\left(\lambda^2\tau^2 + 1\right)\sin(\Phi) + 2\lambda^2\tau\cos(\Phi)\right) + \lambda^2\cos^2(\Phi) - 1, \\ J_6 &= (I_6 - 1) \\ &= \zeta^2\left(\lambda^2\tau^2 + 1\right)\sin^2(\Psi) - 2\zeta\lambda^2\tau\sin(\Psi)\cos(\Psi) + \lambda^2\cos^2(\Psi) - 1. \end{aligned}$$

9.4.1 A cylinder with equal fibers but different orientations

As a first example of the possible behaviors in this rich system, we consider the problem of rotation under pressure in a tube with two fibers with equal strength, $\mu_6 = \mu_4$, but of varying angle. If we fix one fiber, we can vary the angle of the other fiber and ask whether a change in pressure will lead to a reversal in rotation from a left-handed rotation to a right-handed rotation. The condition for inversion is obtained by considering the change of torsion τ with respect to pressure $\tau = \tau(P)$ and identifying the points at which this relationship is stationary. The condition $\tau'(P) = 0$ leads to

$$\mu_4\sin(\Phi - \Psi)\left[6\mu_1(2\cos(\Phi + \Psi) + \cos(3\Phi + \Psi) + \cos(\Phi + 3\Psi)) + 8\mu_4\sin^2(\Phi + \Psi)(\cos(\Phi)\cos(\Psi) - 2\sin(\Phi)\sin(\Psi))\right] = 0. \tag{363}$$

We show in Figure 32 the inversion curves in the parameter space (Φ, Ψ), for $\mu_1 \ll \mu_4$ and $\mu_4 \ll \mu_1$. The two limits are easily obtained analytically from (363) by taking $\mu_1 = 0$ or $\mu_4 = 0$. We see the role of the magic angle introduced in Section 9.3 and discussed below, as being the distinguished value at which an inversion of rotation appears for systems with stiff fibers. In this case, if we start with one fiber oriented at the magic angle and vary the other one there will just be a single inversion of rotation (when the two are equal). For other values, there will be two inversions, one when the two angles are equal and the second at different values of the angle, showing the interesting property of no net rotation (in small deformations) despite the tube being clearly anisotropic.

9.5 Application: The hydrostatic skeleton

A simple cylindrical structure, known as a *cylindrical hydrostatic skeleton* with reinforced fibers is found in many ectothermic and soft-bodied organisms such as nemertean and turbellarian worms. Of particular note is the work of Harris and Crofton [63] and Clark and Cowey [24]. Building on the concept of a hydrostatic skeleton [86, 87, 132], they model the worm body as a cylindrical membrane reinforced with a lattice of crossed, inextensible, fibers winding around the membrane with helical geometry, and they characterize the possible extension and retraction of the lattice network as being analogous to that of lazy tongs or a trellis. Their mathematical model, which is purely geometric, proved to be remarkably successful in explaining their detailed experimental studies of the locomotion and flattening of worms.

Consider a cylinder of height $2\pi P$ and radius R with a helical inextensible fiber of fixed length D as shown in Figure 33. The volume enclosed is simply

$$V = 2\pi^2 R^2 P = D^3/(4\pi)\sin^2\Phi\cos\Phi. \tag{364}$$

The maximal volume is attained when $\partial_\Phi V = 0$, that is for

$$\Phi_{\mathrm{m}}^* = \pi/2 - \tan^{-1}\sqrt{2} \approx 35.26^\circ \tag{365}$$

as shown in Figure 34. For any given volume $0 < V_1 < V_{\max}$, there exist two angles Φ_1 and Φ_2 in $(0, \pi/2)$ that give rise to a cylinder of volume V_1. Therefore, in principle, a worm could change its extension from $H_1 = D\cos\Phi_1$ to $H_2 = D\cos\Phi_2 > H_1$ at constant volume. This change in extension can be obtained by contracting fiber muscles, a transformation that can take place by flattening the cross section to conserve the volume. In the initial and final configurations, the fibers have length D and are at rest. Therefore, in this range of parameters, the hydrostatic skeleton can adopt two shapes of different lengths and maintain these shapes without muscular contraction.

We refer to the reciprocal of this angle, $\Phi_{\mathrm{m}} = \pi/2 - \Phi_{\mathrm{m}}^*$, as the *magic angle* since it seems to appear, as if by magic, in many different settings in mechanics and in solid-state nuclear magnetic resonance [57, 57, 70, 66, 137]. For instance, this angle is believed to be key in understanding the elongation of notochords [1, 89, 90]. Further, an inanimate analogue of the model of Clark and Cowey can be found in the McKibben actuator which consists of a flexible tube surrounded by a sheath of braided families of inextensible fibers helically wound in opposing directions. This design is the basis for so called pneumatic artificial muscles used in robotics, prosthetics, and orthotics [25, 98]. These actuators are typically pressure controlled and their precise functionality is determined by the weave of the fibers. As with the Clark and Cowey model, the fiber winding angle of 35.26° plays a special role in the actuator design.

We will now show that the magic angle also appears naturally as a special limiting case of a nonlinear elastic model. We start our analysis with the simple case of a cylinder with two families of fibers of equal strength ($\mu_6 = \mu_4$) and opposite orientation ($\Psi = \Phi$). This corresponds to the classic case of the McKibben actuators, arteries, and other hydrostats. Under extension or inflation, the couples created by the two fibers cancel out and there is no net couple associated with the deformation and thus no rotation or twist ($\tau = 0$).

We are particularly interested in identifying inversion due to internal change of pressure. We look at the condition under which the radial strain does not vary under a change a pressure. A local analysis for small strains [57] leads to the condition

$$3\mu_1 + \mu_4\sin^2(\Phi)(1 - 3\cos(2\Phi)) = 0. \tag{366}$$

The condition for an inversion in the axial strain is

$$\Phi_{\mathrm{m}}^* = \frac{1}{2}\arccos\left(\frac{1}{3}\right) \approx 35.26440^\circ. \tag{367}$$

By denoting $\mu = \mu_1/\mu_4$ as the ratio of matrix modulus to the fiber modulus, we obtain a complete description of the possible inversions under a change in pressure in the parameter space (μ, Φ) as shown in Figure 35.

Depending on the design criterion, one can consider different tube constructions by varying the fiber angle. For fiber angles larger than Φ_{m} the tube contracts under increased pressure and this behavior provides a model for pneumatic muscles. For tubes with fiber angle close to Φ_{m}, the deformation of the tube in the axial direction is minimal. For fiber angles less than Φ_{m}, the tube extends maximally. Note that this analysis is only valid for small enough P. For larger pressure, we expect the tube to increase eventually in length and radius.

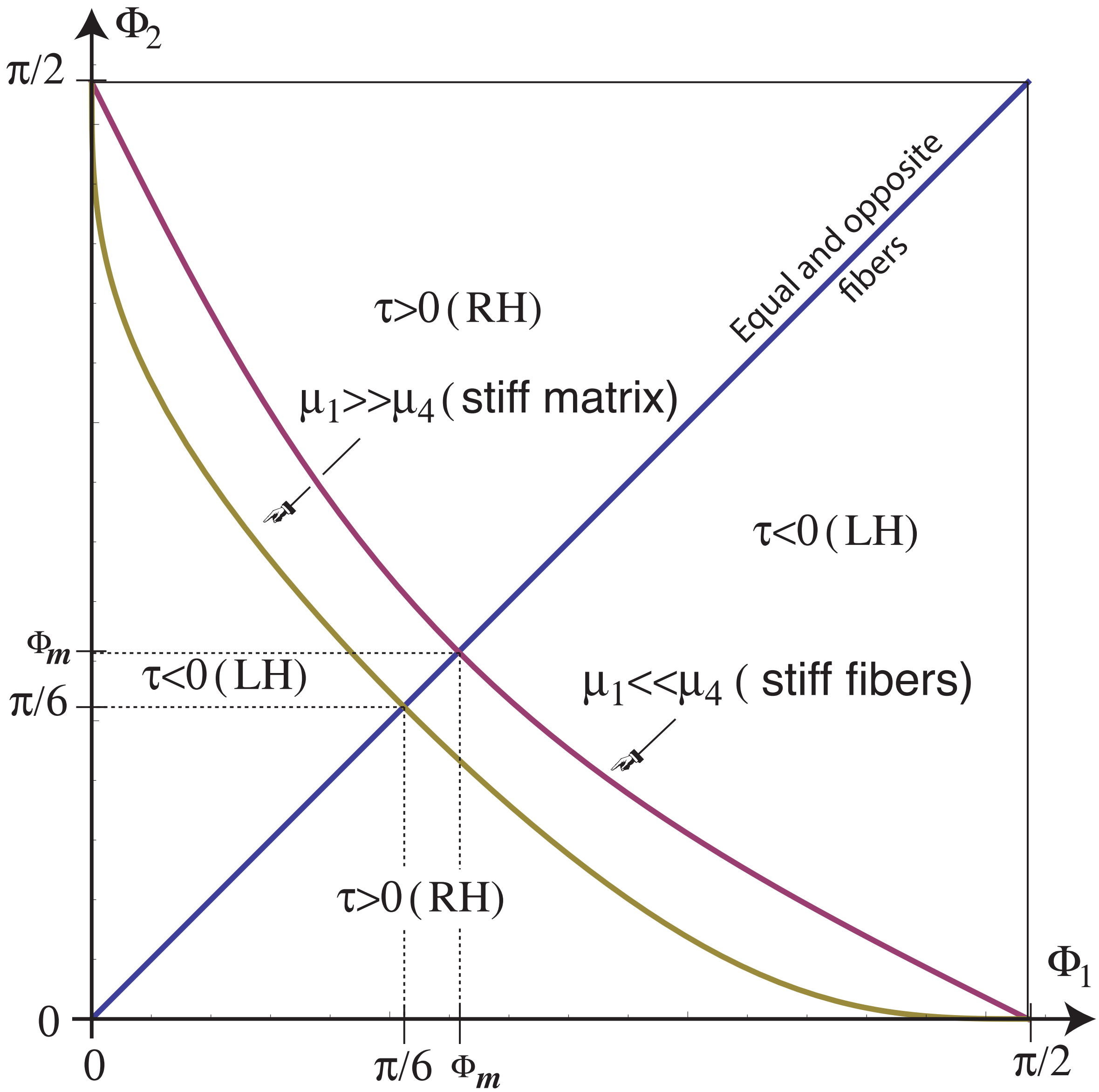


Figure 32: Twist induced by inflation for a cylindrical membrane in small deformation as a function of the orientation of the two fibers.

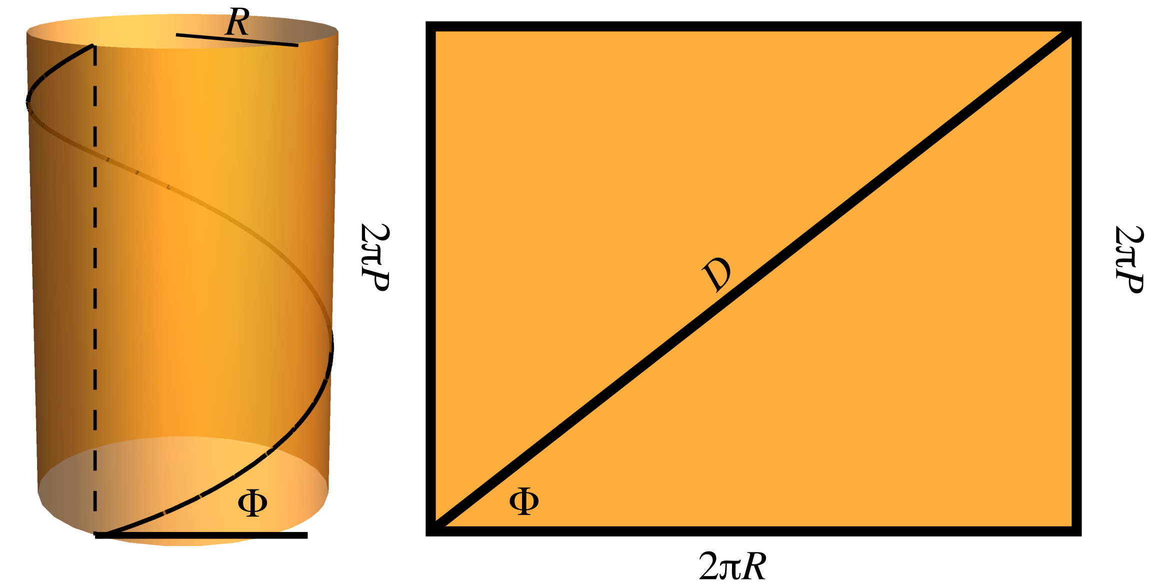


Figure 33: A geometric model of hydrostatic skeleton obtained by assuming that the skeleton can have a cylindrical shape, with inextensible fibers winding helically around the cylinder.

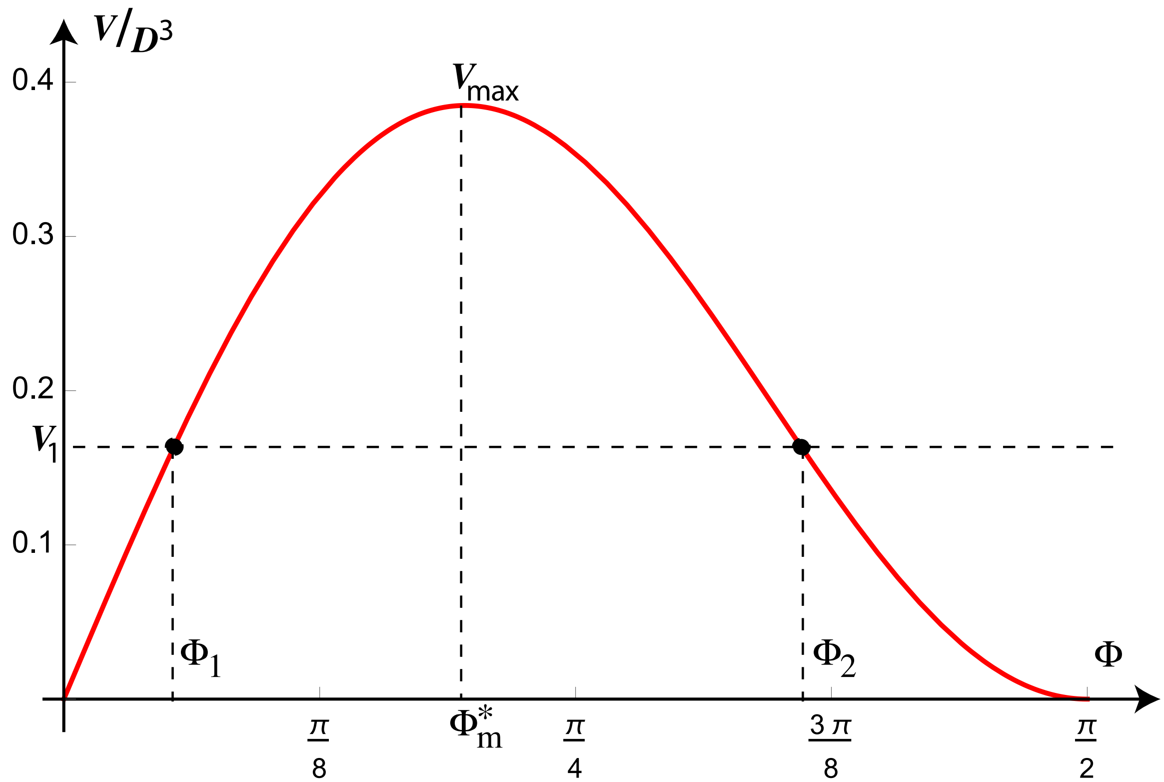


Figure 34: The magic angle gives the maximal enclosed volume of a cylinder reinforced by inextensible fibers of fixed length D. For $V < V_{\max}$ there exist two angles at fixed volume.

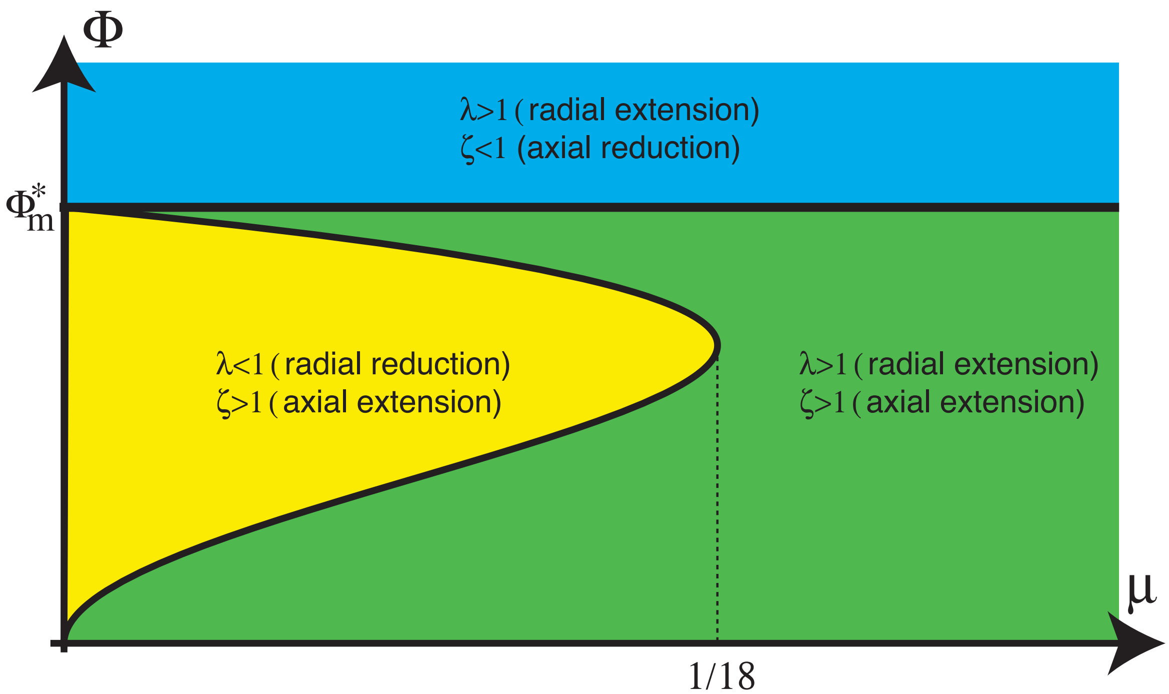


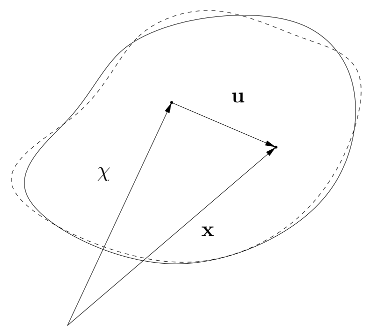
Figure 35: Parameter space for the radial and axial expansion of a thin tube under pressure. Depending on the relative stiffness of the fiber versus the matrix and the fiber angle, a capped tube under pressure can extend radially and axially (bottom left), extend radially but shrink axially (top), or extend axially but shrink radially (bottom right).

10 Linear Elasticity

■ **Overview** We show how to obtain the equations of linear elasticity by linearising the general nonlinear equations of elasticity for small displacements. We also consider the solution of simple problems.

10.1 Infinitesimal strain tensor

The central object is not the mapping $\boldsymbol{\chi}$ but the displacement.



$$\mathbf{u} = \mathbf{x} - \mathbf{X} = \boldsymbol{\chi}(\mathbf{X}, t) - \mathbf{X} \tag{368}$$

$$\implies \quad \nabla \mathbf{u} = \text{Grad } \chi - \mathbb{1} = \mathbf{H} = \mathbf{F} - \mathbb{1}, \tag{369}$$

the displacement gradient.

Assumptions:

- Displacement gradient is small.

Now consider the strain tensor,

$$\mathbf{E}_{\text{nonlin}} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbb{1}\right) \tag{370}$$

$$\mathbf{F} = \mathbb{1} + \mathbf{H} \implies \mathbf{E} = \frac{1}{2}\left((\mathbb{1} + \mathbf{H})\left(\mathbb{1} + \mathbf{H}^T\right) - \mathbb{1}\right) = \underbrace{\frac{1}{2}(\mathbf{H} + \mathbf{H}^T)}_{\mathbf{E}} + \mathcal{O}(\mathbf{H}^2) \tag{371}$$

$$\mathbf{E} = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T) = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right) \tag{372}$$

10.2 Constant relationships

$$\mathbf{S} = \mathcal{S}(\mathbf{F}), \qquad \mathbf{T} = \mathcal{T}(\mathbf{F}). \tag{373}$$

We assume $\mathcal{S}(\mathbb{1}) = 0$ (no residual stress).

$$\implies \mathbf{S} = \mathcal{S}(\mathbb{1} + \mathbf{H}) = \underbrace{\mathcal{S}(\mathbb{1})}_{0} + \underbrace{D\mathcal{S}(\mathbb{1})[\mathbf{H}]}_{C[\mathbf{H}]} + \mathcal{O}(\mathbf{H}^2), \tag{374}$$

where C is linear in $\mathbf{H}$.

$$\mathbf{T} = \underbrace{\mathcal{T}(\mathbb{1})}_{0} + D\mathcal{T}(\mathbb{1})[\mathbf{H}] + \mathcal{O}(\mathbf{H}^2) \tag{375}$$

Which one to use?

$$\mathbf{T} = J^{-1}\mathbf{F}\mathbf{S} \tag{376}$$

$$\implies \mathcal{T} = J^{-1}\mathbf{F}\mathcal{S} \tag{377}$$

and

$$\mathcal{T} = D\mathcal{T}[\mathbf{H}] = J^{-1}(\mathbb{1} + \mathbf{H})D\mathcal{S}(\mathbb{1})[\mathbf{H}] = \det(\mathbb{1} + \mathbf{H})(\mathbb{1} + \mathbf{H})D\mathcal{S}(\mathbb{1})[\mathbf{H}] + \mathcal{O}(\mathbf{H}^2) \tag{378}$$

$$\implies D\mathcal{T}[\mathbf{H}] = D\mathcal{S}[\mathbf{H}] = D\mathcal{S}[\mathbf{H}] = C[\mathbf{H}] \tag{379}$$

C elasticity tensor,

$$T_{ij} = C_{ijk\ell}H_{k\ell} \tag{380}$$

Major symmetries:

$$C_{ijk\ell} = C_{k\ell ij} \tag{381}$$

Minor symmetries:

$$C_{ijk\ell} = C_{ij\ell k} = C_{jik\ell} = C_{ji\ell k} \tag{382}$$

$\implies$ from 81 components to 36 independent components.

Note also

$$T_{ij} = C_{ijk\ell}\Big[\underbrace{\frac{1}{2}(\mathbf{H} + \mathbf{H}^T)}_{\mathbf{E}} + \frac{1}{2}(\mathbf{H} - \mathbf{H}^T)\Big]_{k\ell} \tag{383}$$

$$\boxed{T_{ij} = C_{ijk\ell}\mathbf{E}_{k\ell}} \tag{384}$$

constant relationship for linear elasticity.

10.3 Isotropic linear elasticity

If the material is isotropic:

$$S_{ij} = T_{ij} = 2\mu e_{ij} + \lambda(\mathrm{tr}\ \mathbf{E})\delta_{ij} \tag{385}$$

where

$$C_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + 2\mu\delta_{ik}\delta_{j\ell} \tag{386}$$

for μ and λ the Lamé coefficients. From the symmetry of C and positive definiteness, we have

$$\mu > 0, \qquad 2\mu + 3\lambda > 0. \tag{387}$$

Note: C is positive definite means

$$\mathbf{E}.\mathsf{C}(\mathbf{E}) > 0, \qquad \forall\mathbf{E} \in \mathrm{Sym}. \tag{388}$$

Material hyperelasticity $\iff$ C is positive definite
$\implies$ C is symmetric.

If the body is *homogeneous*, then ρ_0, λ, μ are constant.

10.3.1 Equations:

$\mathbf{u} = (\chi)(\mathbf{X}) - \mathbf{X}$.

$$\mathbf{S} = \mathsf{C}[\mathbf{E}], \qquad e = \tfrac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) \tag{389}$$

$$\text{Div } \mathbf{S} + \mathbf{b}_0 = \rho_0\ddot{\mathbf{u}} = \rho_0\mathbf{a} \tag{390}$$

Assume homogeneity and isotropy,

$$\mathbf{S} = 2\mu\mathbf{E} + \lambda(\text{tr } \mathbf{E})\mathbb{1}. \tag{391}$$

$$\begin{aligned}
\text{Div } \mathbf{S} &= \not{2}\mu\text{Div}\left(\frac{1}{\not{2}}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)\right) + \lambda\text{Div}\left(\text{tr } \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)\mathbb{1}\right) && (392)\\
&= \mu\Delta\mathbf{u} + \mu\text{Grad Div } \mathbf{u} + \lambda\text{Grad Div } \mathbf{u} && (393)\\
&= \mu\Delta\mathbf{u} + (\mu + \lambda)\text{Grad Div } \mathbf{u} && (394)
\end{aligned}$$

Therefore we have the *Navier equation*,

$$\boxed{\mu\Delta\mathbf{u} + (\mu + \lambda)\text{Grad Div } \mathbf{u} + \mathbf{b}_0 = \rho_0\ddot{\mathbf{u}}} \tag{395}$$

Note that $\mathbf{u} = \mathbf{u}(\mathbf{X})$ implies that $\mathbf{x}$ does not appear any more (we can replace $\mathbf{X}$ by $\mathbf{x}$ if we want – I don't).

In components,

$$\mathbf{u} = \mathbf{u}(\mathbf{X}, t) = u_i\mathbf{E}_i \tag{396}$$

implies

$$\boxed{\rho_0\frac{\partial^2 u_i}{\partial t^2} = b_i + \mu\sum_{j=1}^{3}\frac{\partial^2 u_i}{\partial X_j^2} + (\lambda + \mu)\frac{\partial^2 u_j}{\partial X_i \partial X_j}} \tag{397}$$

10.4 Examples

To understand the meaning of the elastic moduli, we consider simple deformations.

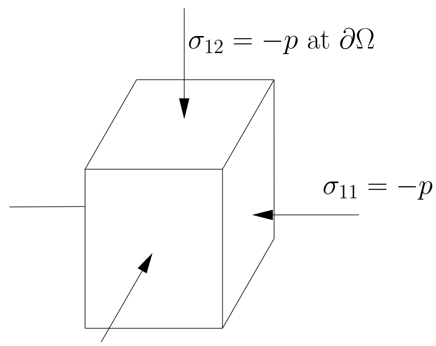
1) Pure shear, $\mathbf{u} = (\gamma X_2, 0, 0)$

$$[\mathbf{E}] = \frac{1}{2}\begin{bmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad [\sigma] = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{398}$$

$\implies \tau = \mu\gamma \implies \mu$ is the shear modulus.

2) Uniform compression, $\mathbf{u} = \delta\mathbf{X}$ and $\mathbf{u} = \mathbf{x} - \mathbf{X} = (\delta + 1)\mathbf{X} - \mathbf{X}$

$$\mathbf{E} = \delta\mathbb{1}, \qquad \sigma = -p\mathbb{1} \tag{399}$$



We use

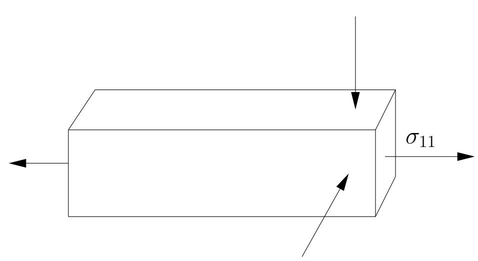
$$\mathbf{E} = \frac{1}{2\mu}\left[\sigma - \frac{\lambda}{2\mu+3\lambda}(\mathrm{tr}\ \sigma)\mathbb{1}\right] \tag{400}$$

$$\begin{aligned} \delta\mathbb{1} &= \frac{1}{2\mu}\left[-p\mathbb{1} + \frac{\lambda}{2\mu+3\lambda}3p\mathbb{1}\right] &(401)\\ &= \frac{1}{2\mu}p\left[\frac{-(2\mu+3\lambda)+3\lambda}{2\mu+3\lambda}\right]\mathbb{1} &(402)\\ &= -\frac{p}{2\mu+3\lambda} &(403)\end{aligned}$$

$$\implies p = -(2\mu+3\lambda)\delta = -3\underbrace{\left(\frac{2\mu+3\lambda}{3}\right)}_{\kappa}\delta, \tag{404}$$

where κ is the *modulus of compression*. Remember the condition $2\mu + 3\lambda > 0$!

3) Uniaxial tension, $\sigma = t\mathbf{E}_1 \otimes \mathbf{E}_1$



$$[\mathbf{E}] = \mathrm{diag}(\alpha, \beta, \beta), \qquad \alpha = \frac{t}{E}, \qquad \beta = -\nu\alpha. \tag{405}$$

$$E = \frac{\mu(2\mu+3\lambda)}{\mu+\lambda}, \qquad \nu = \frac{\lambda}{2(\mu+\lambda)} \tag{406}$$

Here E is equated to the *infinitesimal Young's modulus* and ν is equated to *Poisson's ratio.*

$$\boxed{\mathbf{E} = \frac{1}{E}((1+\nu)\sigma - \nu(\mathrm{tr}\ \ \sigma)\mathbb{1})} \tag{407}$$

an alternative form for $\mathbf{E}$.

Expect $\nu > 0$ (Care! It's an auxetic material!)

Now

$$\kappa = \frac{2\mu + 3\lambda}{3} = \frac{E}{3(1-2\nu)}, \tag{408}$$

so that as $\nu \to 1/2$, $\kappa \to \infty$, and we would need an infinite force to change the volume. Therefore incompressible materials have $\nu = 1/2$.

10.5 Incompressible linear elasticity

Recall: Incompressibility:

$$\det \mathbf{F} = 1, \qquad \Longrightarrow \det(\mathbb{1} + \mathbf{H}) = 1 + \text{tr } \mathbf{H} + \mathcal{O}(\mathbf{H}^2) = 1 \tag{409}$$

Therefore tr $\mathbf{H} = 0 = \text{Div } \mathbf{u}$, and

$$\boxed{\text{Div } \mathbf{u} = 0} \iff \boxed{\text{tr } \mathbf{E} = 0} \tag{410}$$

Also

$$\mathbf{T} = -p\mathbb{1} + J^{-1}\mathbf{F}\frac{\partial \mathsf{W}}{\partial \mathbf{F}}, \qquad \sigma = -p\mathbb{1} + C_{ijk\ell}e_{k\ell} \tag{411}$$

For isotropic material,

$$\sigma = 2\mu\mathbf{E} + \cancel{\lambda(\text{tr } \mathbf{E})}\mathbb{1} - p\mathbb{1} \tag{412}$$

but

$$\mu = \frac{E}{2(1+\nu)} = \frac{E}{3}. \tag{413}$$

Therefore

$$\boxed{\rho\ddot{\mathbf{u}} = -\text{Grad } p + \mu\Delta\mathbf{u}} \tag{414}$$

and

$$\boxed{\mu = \frac{E}{3}} \tag{415}$$

N.B. Boundary conditions

$$\mathbf{u} = \mathbf{u}^*(t), \quad \text{on } \partial_1\mathcal{B} \text{ displacements} \tag{416}$$

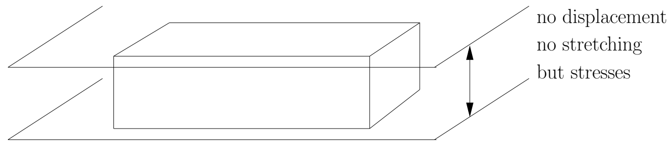
$$\tau\mathbf{n} = \mathbf{T}^*(t), \quad \text{on } \partial_2\mathcal{B} \text{ traction} \tag{417}$$

10.5.1 General principles

1) Linear superposition

2) Stresses, strains and displacements are proportional to the loads (or displacements) applied to the solid.

3) If $\partial_2\mathcal{B} = \emptyset$ then there exists one unique solution, only displacements.



4) If only traction and tractions are in equilibrium, then stresses and strains are unique. For initial conditions, there exists one unique $u(t)$.

Some nomenclature about loading

1) Plain strain

$$\mathbf{u} = (u(X,Y), v(X,Y), 0) \implies e_{13} = e_{23} = e_{33} = 0, \qquad \tau_{13} = \tau_{23} = \tau_{31} = \tau_{32} = 0. \tag{418}$$

2) Plane stress

$$\tau_{13} = \tau_{23} = \tau_{33} = 0, \qquad \tau = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{419}$$

3) Antiplane strain

$$\mathbf{u} = (0, 0, w(X,Y)) \tag{420}$$

4) Pure torsion

$$\mathbf{u} = (-\Omega YZ, \Omega XZ, \Omega\varphi(X,Y)) \tag{421}$$

(see problem sheet 6)

10.6 Plane/Strain/Stress Solutions

10.6.1 Plane solutions

(stress or strains)

In cartesian coordinates,

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} = \frac{1+\nu}{E}\tau_{ij} - \frac{\nu}{E}\tau_{kk}\delta_{ij} \tag{422}$$

$$\frac{\partial \tau_{ij}}{\partial x_i} + b_j = 0 \tag{423}$$

Assume that b derives from a potential,

$$b_i = \frac{\partial V}{\partial X_i}, \quad i = 1, 2, \quad b_3 = 0. \tag{424}$$

Plane stresses or strain $\tau_{13} = \tau_{23} = 0$.

10.6.2 Idea

Let

$$\tau_{11} = \frac{\partial \phi}{\partial X_2} - V, \qquad \tau_{22} = \frac{\partial \phi}{\partial X_1} - V, \qquad \tau_{12} = -\frac{\partial^2 \phi}{\partial X_1 \partial X_2}, \qquad \tau_{33} = \beta\nu(\tau_{11} + \tau_{22}), \tag{425}$$

$\beta = 0$ is plane stress and $\beta = 1$ is plane strain.

10.6.3 Equations

$$\frac{\partial \tau_{11}}{\partial X_1} + \frac{\partial \tau_{12}}{\partial X_2} + b_1 = 0, \qquad \frac{\partial \tau_{12}}{\partial X_1} + \frac{\partial \tau_{22}}{\partial X_2} + b_2 = 0. \tag{426}$$

Therefore

$$\frac{\partial}{\partial X_1}\left(\frac{\partial^2 \phi}{\partial X_2^2} - \cancel{V}\right) + \frac{\partial}{\partial X_2}\left(-\frac{\partial^2 \phi}{\partial X_1 \partial X_2}\right) + \cancel{b_1} = 0, \tag{427}$$

$$\frac{\partial}{\partial X_1}\left(-\frac{\partial^2 \phi}{\partial X_1 \partial X_2}\right) + \frac{\partial}{\partial X_2}\left(\frac{\partial^2 \phi}{\partial X_2^2} - \cancel{V}\right) + \cancel{b_2} = 0, \tag{428}$$

and the equations of motion are satisfied. But we do not have an equation for ϕ. We have equations for τ_{ij} or e_{ij}, that is, 6 fields but u_i is 3 components.

10.6.4 Compatibility conditions

Recall: conditions for $\mathbf{F}$: Curl $\mathbf{F} = 0$. For

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{429}$$

Compatibility conditions:

$$\text{Curl Curl } \mathbf{E} = 0, \tag{430}$$

$$\iff \epsilon_{ipm}\epsilon_{jqn}\frac{\partial^2 e_{mn}}{\partial X_p \partial X_q} = 0 \tag{431}$$

$$\iff \frac{\partial^2 e_{ij}}{\partial X_k \partial X_\ell} + \frac{\partial^2 e_{k\ell}}{\partial X_i \partial X_j} - \frac{\partial^2 e_{i\ell}}{\partial X_j \partial X_k} - \frac{\partial^2 e_{jk}}{\partial X_i \partial X_\ell} = 0 \tag{432}$$

These are <u>6</u> relations (but only 3 are independent). For planar problems: $e_{13} = e_{23} = 0$, $\partial e_{ij}/\partial X_3 = 0$,

$$\implies \frac{\partial^2 e_{11}}{\partial X_2^2} + \frac{\partial^2 e_{22}}{\partial X_1^2} - 2\frac{\partial^2 e_{12}}{\partial X_1 \partial X_2} = 0. \tag{433}$$

Now for plane stress we have $\tau_{33} = 0$ and from plane strain we have $\tau_{33} = \nu(\tau_{11} + \tau_{22})$,

$$\iff \tau_{33} = \beta\nu(\tau_{11} + \tau_{22}), \tag{434}$$

which implies

$$e_{11} = \frac{1+\nu}{E}\tau_{11} - \frac{\nu}{E}(1+\beta\nu)(\tau_{11} + \tau_{22}) \tag{435}$$

$$e_{22} = \frac{1+\nu}{E}\tau_{22} - \frac{\nu}{E}(1+\beta\nu)(\tau_{11} + \tau_{22}) \tag{436}$$

$$e_{12} = \frac{1+\nu}{E}\tau_{12} \tag{437}$$

Insert these into (∗) and use $\tau_{11} = \frac{\partial^2 \phi}{\partial X_1^2} - V$,

$$\implies \frac{\partial^4 \phi}{\partial x_1^4} + 2\frac{\partial^4 \phi}{\partial x_1^2 \partial_2^2} + \frac{\partial^4 \phi}{\partial x_2^4} = \frac{1 - \beta\nu^2}{1 - \nu - 2\beta\nu^2}\left(\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2}\right) \tag{438}$$

$$\iff \boxed{\nabla^4\phi = C_\nu \Delta V}, \qquad C_\nu = \frac{1-\beta\nu^2}{1-\nu-2\beta\nu^2}. \tag{439}$$

Here ∇^4 is the *biharmonic operator* and ϕ is the *Airy potential.* If $\beta = 0$, we have plane stress and $\beta = 1$ is plane strain.

10.6.5 Application

10.7 Elasto-dynamics

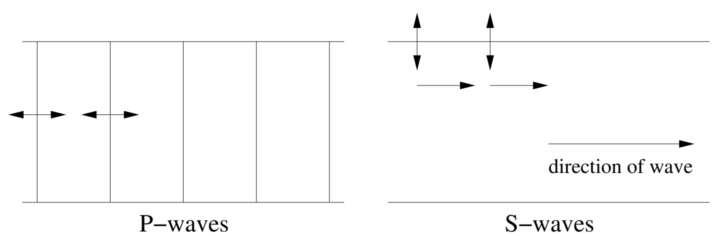
$$\mu\Delta\mathbf{u} + (\lambda+\mu)\text{Grad Div }\mathbf{u} = \rho\ddot{\mathbf{u}} \tag{$*$}$$

10.7.1 Planar waves

$$\mathbf{u}(\mathbf{x},t) = \mathbf{a}\sin(\mathbf{k}\cdot\mathbf{x} - ct) \tag{440}$$

Here $\mathbf{a}$ is the amplitude, $\mathbf{k}$ is the direction and c is the velocity. We normalize such that $|\mathbf{k}| = 1$.
2 interesting cases:

$\mathbf{a} \parallel \mathbf{k}$ longitudinal – primary, pressure, P-waves.
$\mathbf{a} \perp \mathbf{k}$ transverse – shear, secondary, S-waves.



Let $\varphi(\mathbf{x},t) = \mathbf{k}\cdot\mathbf{x} - ct$.

$$\text{Note that Div }\mathbf{u} = \mathbf{a}\cdot\mathbf{k}\cos\varphi \tag{441}$$
$$\text{Curl }\mathbf{u} = -\mathbf{a}\times\mathbf{k}\cos\varphi \tag{442}$$

Div $\mathbf{u} = 0$ is transverse, Curl $\mathbf{u} = 0$ is longitudinal.
Substitute $\mathbf{u} = \mathbf{a}\sin\varphi$ in ($*$). Then

$$\Delta\mathbf{u} = -\mathbf{a}\sin\varphi \tag{443}$$
$$\text{Grad Div }\mathbf{u} = \text{Grad}(\mathbf{a}\cdot\mathbf{k}\cos\varphi) = (\mathbf{a}\cdot\mathbf{k})\mathbf{k}(-\sin\varphi) \tag{444}$$

Therefore $\ddot{\mathbf{u}} = -c^2\mathbf{a}\sin\varphi$ and

$$\mu\mathbf{a} + (\lambda+\mu)(\mathbf{a}\cdot\mathbf{k})\mathbf{k} = \rho c^2\mathbf{a} \tag{445}$$

This is a linear operator on $\mathbf{a}$. Define A the *acoustic tensor*,

$$\mathsf{A} = \frac{1}{\rho}\left(\mu\mathbb{1} + (\lambda+\mu)\mathbf{k}\otimes\mathbf{k}\right)[\mathsf{A}]_{ij} = \frac{1}{\rho}\left(\mu\delta_{ij} + (\lambda+\mu)k_ik_j\right) \tag{446}$$

so that we have the eigenvalue problem

$$\mathsf{A}\mathbf{a} = c^2\mathbf{a} \tag{447}$$

1) $\mathbf{a} = \alpha\mathbf{k}$

$$\alpha A_{ij}k_j = \frac{1}{\rho}(\mu k_i + (\lambda+\mu)\underbrace{k_j k_j}_{1} k_i) = c^2 \alpha k_i \tag{448}$$

$$\implies \frac{\lambda + 2\mu}{\rho} = c^2, \qquad c_L = \sqrt{\frac{\lambda+2\mu}{\rho}} \tag{449}$$

2) $\mathbf{a} \perp \mathbf{k}$, $a_i k_i = 0$.

$$A_{ij}a_j = \frac{1}{\rho}\left(\mu a_i + (\lambda+\mu)k_i k_j a_j\right) = c^2 a_i \tag{450}$$

$$\implies c^2 = \frac{\mu}{\rho}, \qquad c_T = \sqrt{\frac{\mu}{\rho}} < \sqrt{\frac{\lambda+2\mu}{\rho}}, \tag{451}$$

i.e. slower than c_L.

Note also

$$c_L = \sqrt{\frac{E(1-\nu)}{(1+\nu)(1-2\nu)\rho}}, \tag{452}$$

where $1-2\nu = 0$ for an incompressible material. Therefore $c_L \to \infty$ as $\nu \to 1/2$.

Also note

$$c_T^2 = \mu/\rho \qquad \implies \qquad \mu = c_T^2 \rho \tag{453}$$

$$c_L^2 = \frac{\lambda}{\rho} + 2c_T^2, \qquad \implies \qquad \rho c_L^2 - 2\rho c_T^2 \tag{454}$$

$$\implies \boxed{\ddot{\mathbf{u}} = c_T^2 \Delta \mathbf{u} + (c_L^2 - c_T^2)\text{Grad Div } \mathbf{u}} \tag{455}$$

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