C4.8 Complex Analysis: conformal maps and geometry

Sheet 4

Problem 1.

In the lectures I gave a sketch of the proof of the Distortion Theorem (Theorem 3.2.9 in the lecture notes). Write the complete proof of this theorem. (There is no need to write the "moreover" part.)

Problem 2.

The harmonic measure is conformally invariant by the definition. Let us assume that the boundary $\partial\Omega$ is smooth. In this case the harmonic measure $\omega(z, A)$ is continuous with respect to the arc-length i.e. there is the density function $h_z(\zeta) = h_{z,\Omega}(\zeta)$ on the boundary of Ω such that

$$\omega(z, A) = \int_A h_z(\zeta)\mathrm{d}s(\zeta)$$

where $\mathrm{d}s$ is the arc-length.

(1) Let Ω and Ω' be two simply connected domains with analytic boundary, so that the Riemann maps are differentiable on the boundary. Let $f : \Omega \to \Omega'$ be a conformal transformation. Derive the relation between $h_{z,\Omega}(\zeta)$ and $h_{f(z),\Omega'}(f(\zeta))$.

(2) Let $\Omega = \mathbb{D}$, compute the density of harmonic measure with the pole at $z_0 \in \mathbb{D}$.

(3) Show that the density of the harmonic measure $h_{z,\Omega}(\zeta)$ is equal to $\partial_n G_\Omega(z_0, \zeta)/2\pi$, where $\partial_n G$ is the normal derivative of the Green's function on the boundary.

(4) Use the connection between the Green's function and the harmonic measure to derive the result from (1)

(5) (Bonus question) Have you seen the function h before? What is the name for this function?

Problem 3.

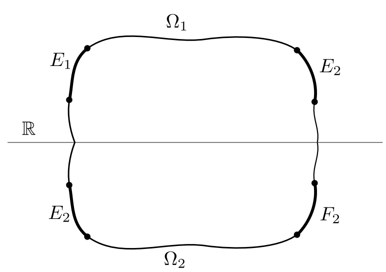
Let Γ be the family of rectifiable curves in the annulus $A(r, R)$ that are not contractable, that is go around the circle $|z| = r$ and let Γ' be the family of rectifiable curves in $A(r, R)$ that connect two boundary components. Find $\lambda(\Gamma)$ and $\lambda(\Gamma')$.

Problem 4.

Use the symmetry rule to prove the following statement.

Let Ω_1 be a domain in the upper half plane and let E_1 and F_1 be two sets on $\partial\Omega$. Let Ω_2, E_2, and F_2 be their symmetric images with respect to $\mathbb{R}$. We define $\Omega = \Omega_1 \cup \Omega_2$ (to be completely rigorous we also have to add the real part of the boundary), $E = E_1 \cup E_2$, and $F = F_1 \cup F_2$. Then

$$2d_\Omega(E, F) = d_{\Omega_1}(E_1, F_1) = d_{\Omega_2}(E_2, F_2).$$



Problem 5.

Let Ω be a simply connected domain, $z_0 \in \Omega$ and A be an arc (connected set) on the boundary of Ω. If you wish, you may assume that Ω is a nice domain, say, a domain bounded by an analytic Jordan curve, but this is not too important.

(1) Let Ω', z_0' and A' be another domain, a point and an arc as above. Show that there is a conformal map f such that $f(\Omega) = \Omega'$, $f(z_0) = z_0'$ and $f(A) = A'$ if and only if $\omega_\Omega(z_0, A) = \omega_{\Omega'}(z_0', A')$.

(2) Let Γ be the family of all rectifiable curves in Ω such that their endpoints are on A and they separate z_0 from $\partial\Omega \setminus A$. Show that there is a function F (independent of Ω, z_0 and A) such that $\lambda(\Gamma) = F(\omega_\Omega(z_0, A))$.

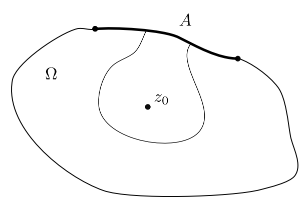


FIGURE 1. Family of curves Γ.

(3) By part (1) we can assume without loss of generality that $\Omega = \mathbb{D}$, $z_0 = 0$ and A is the arc $\{e^{i\theta}, -\theta_0 < \theta < \theta_0\}$ for some $\theta_0 \in [0, \pi)$. Let Γ be the family of curves as defined in part (2). Show that

$$\lambda(\Gamma) = 2d_{\mathbb{D}_+}([-1, 0], A_+) = 4\mathrm{d}_{\mathbb{D}\setminus[-1,0]}([-1, 0], A),$$

where $\mathbb{D}_+$ is the upper half-disc, A_+ is the upper half of A and $d_\Omega(E, F)$ is the extremal distance, that is the extremal length of the family of curves connecting boundary sets E and F inside Ω.

(4) Our next goal is to compute $d_{\mathbb{D}_+}([-1, 0], A_+)$. We know that $\mathbb{D}_+$ with marked points $-1, 0, 1, e^{i\theta_0}$ could be mapped onto a rectangle in such a way that the marked points are mapped to the vertices. Use this fact to compute $d_{\mathbb{D}_+}([-1, 0], A_+)$ in terms of θ_0. Combine all the results to find a formula for the function F from part (2).

(Hint: Use the fact that the upper half-plane with marked points $-1/k, -1, 1, 1/k$ could be mapped onto a rectangle with the ratio of side lengths equal to $2K(k)/K'(k)$, where K and K' are the complete elliptic integral of the the complementary complete elliptic integrals of the first king. You don't need to know anything about K and K', the only important thing is that they give an explicit expression for the side length ratio in terms of k.)

Problem 6.

Let Ω be a conformal triangle, i.e. a simply connected domain bounded by a Jordan curve with three marked points on it. We will call these marked points the vertices and the arcs between them the sides of the conformal triangle Ω. Let γ be a continuous curve in $\bar{\Omega}$ such that it intersects with all three sides of Ω.

Use extremal lengths to show that there exists a curve γ as above such that

$$L(\gamma) \leq 3^{1/4}\sqrt{A(\Omega)}$$

where $L(\gamma)$ is the usual Euclidean length of γ and $A(\Omega)$ is the area of Ω.

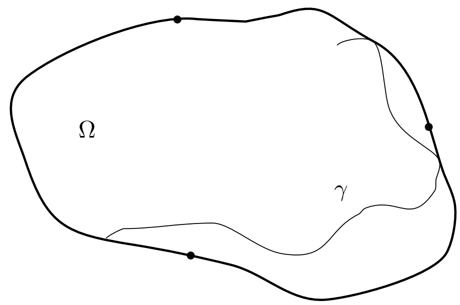


FIGURE 2. Conformal triangle and a curve inside touching all three sides.

Show that the constant $3^{1/4}$ is sharp.