C4.8 Complex Analysis: conformal maps and geometry

Sheet 4

Problem 1.

In the lectures I gave a sketch of the proof of the Distortion Theorem (Theorem 3.2.9 in the lecture notes). Write the complete proof of this theorem. (There is no need to write the "moreover" part.)

Solution. First of all we verify $zf''(z)/f'(z) = r\partial_r \log f'$. Indeed

$$r\partial_r \log f'(re^{i\theta}) = r\frac{\partial_r f'(re^{i\theta})}{f'(re^{i\theta})} = \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})} = \frac{zf''(z)}{f'(z)}$$

So

$$\frac{-4r+2r^2}{1-r^2} \le \Re\left(\frac{zf''(z)}{f'(z)}\right) \le \frac{4r+2r^2}{1-r^2}$$

immediately implies

$$\frac{-4+2r}{1-r^2} \le \partial_r \ln|f'(z)| \le \frac{4+2r}{1-r^2}$$

Integrating with respect to r we have

$$\log|f'(re^{i\theta})| = \log|f'(re^{i\theta})| - \log|f'(0)| = \int_0^r \partial_r \ln|f'(re^{i\theta})| \mathrm{d}r$$

Now we can plug in the above inequality into the integral to get

$$\int_0^r \frac{-4+2r}{1-r^2}\mathrm{d}r \le \log|f'(re^{i\theta})| \le \int_0^r \frac{4+2r}{1-r^2}\mathrm{d}r$$

Explicit integration gives

$$\log\left(\frac{1-r}{(1+r)^3}\right) \le \log|f'(re^{i\theta})| \le \log\left(\frac{1+r}{(1-r)^3}\right)$$

which, after exponentiation, gives the Distortion theorem. □

Problem 2.

The harmonic measure is conformally invariant by the definition. Let us assume that the boundary $\partial\Omega$ is smooth. In this case the harmonic measure $\omega(z, A)$ is continuous with respect to the arc-length i.e. there is the density function $h_z(\zeta) = h_{z,\Omega}(\zeta)$ on the boundary of Ω such that

$$\omega(z, A) = \int_A h_z(\zeta)\mathrm{d}s(\zeta)$$

where $\mathrm{d}s$ is the arc-length.

(1) Let Ω and Ω' be two simply connected domains with analytic boundary, so that the Riemann maps are differentiable on the boundary. Let $f : \Omega \to \Omega'$ be a conformal transformation. Derive the relation between $h_{z,\Omega}(\zeta)$ and $h_{f(z),\Omega'}(f(\zeta))$.

(2) Let $\Omega = \mathbb{D}$, compute the density of harmonic measure with the pole at $z_0 \in \mathbb{D}$.

(3) Show that the density of the harmonic measure $h_{z,\Omega}(\zeta)$ is equal to $\partial_n G_\Omega(z_0, \zeta)/2\pi$, where $\partial_n G$ is the normal derivative of the Green's function on the boundary.

(4) Use the connection between the Green's function and the harmonic measure to derive the result from (1)

(5) (Bonus question) Have you seen the function h before? What is the name for this function?

Solution. (1) We know that harmonic measure is conformally invariant

$$\int_A h_{z_0,\Omega}(\zeta)\mathrm{d}s(\zeta) = \omega_\Omega(z_0, A) = \omega_{\Omega'}(f(z_0), f(A)) = \int_{f(A)} h_{f(z_0),\Omega'}(\zeta)\mathrm{d}s(\zeta)$$

changing variables in the last integral to $\zeta = f(t)$ for which we have $\mathrm{d}s(\zeta) = |f'(t)|\mathrm{d}s(t)$ we get

$$\int_A h_{z_0,\Omega}(\zeta)\mathrm{d}s(\zeta) = \int_A h_{f(z_0),\Omega'}(f(t))|f'(t)|\mathrm{d}s(t)$$

Since this is true for every A, we have that

$$h_{z_0,\Omega}(\zeta) = |f'(t)|h_{f(z_0),\Omega'}(f(t)).$$

This means that the density of harmonic measure is covariant, which should not be a surprise.

(2) Let $f(z) = (z - z_0)/(1 - \bar{z}_0 z)$ be a conformal transformation of $\mathbb{D}$ which sends z_0 to 0. By the formula from the previous part and writing θ instead of $e^{i\theta}$ we have

$$h_{z_0}(e^{i\theta}) = |f'(e^{i\theta})|h_0(f(e^{i\theta})) = \frac{1}{2\pi}|f'(e^{i\theta})|$$

where the last equality follows from the definition of harmonic measure in $\mathbb{D}$. Differentiating the formula for f we get

$$|f'(\zeta)| = \frac{1 - |z_0|^2}{|1 - \bar{z}_0 z|^2} = \frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2}$$

where $\zeta = e^{i\theta}$ and $z_0 = re^{i\phi}$.

(3) There are many ways to prove this, one of the standard ways is to use Green's formula. I think that the easiest on is to use the conformal invariance.

In the case $\Omega = \mathbb{D}$ and $z_0 = 0$ this is easy to check: $G(re^{i\theta}) = -\log|z| = -\log(r)$. The normal derivative on the boundary is $1/r = 1$. Since the density of harmonic measure is $1/2\pi$ we have de desired result.

For other domains we notice that the Green's function is conformally invariant $G_\Omega(z_0, z) = G_{f(\Omega)}(f(z_0), f(z))$, hence its normal derivative is covariant

$$\partial_n G_\Omega(z_0, \zeta) = |f'(\zeta)|\partial_n G_{f(\Omega)}(f(z_0), f(\zeta)).$$

(4) The argument above works in both directions. If we know that h is given by the normal derivative of G (which is covariant since G is invariant), we immediately have that h is also covariant.

This might look by a circular argument, but fortunately there is a way to prove (3) without conformal invariance. From Green's second identity where one of the functions is the Green's function and the other is harmonic we can get the following formula for the solution of Dirichlet boundary problem

$$u(z_0) = \int_{\partial\Omega} u(\zeta)\partial_n G(z_0, \zeta)\mathrm{d}s(\zeta)$$

(here it is important to notice that the Green's function in this formula differs from ours exactly by the factor of 2π.

(5) The function h is the Poisosn kernel

$\square$

Problem 3.

Let Γ be the family of rectifiable curves in the annulus $A(r,R)$ that are not contractable, that is go around the circle $|z|=r$ and let Γ' be the family of rectifiable curves in $A(r,R)$ that connect two boundary components. Find $\lambda(\Gamma)$ and $\lambda(\Gamma')$.

Solution. Let us consider the metric $\rho(z)=1/|z|$. We are going to show that this metric is extremal.

Let Γ_0 be the family of all circles with center at the origin. By γ_{r_0} we denote the circle with radius r_0. It is easy to see that $L(\gamma_{r_0},\rho)=\int_{\gamma_{r_0}} r_0^{-1}\mathrm{d}s=\int_0^{2\pi}\mathrm{d}\theta=2\pi$.

For an arbitrary $\gamma\in\Gamma$, since γ goes around the origin, for each θ there is at least one point of γ with argument θ, let us denote this point by $\gamma(\theta)$. Then

$$L(\gamma,\rho)=\int_\gamma \frac{1}{|\gamma|}\mathrm{d}s\geq\int_0^{2\pi}=2\pi.$$

This proves that the curves of Γ_0 are the shortest curves in Γ with respect to our metric ρ.

Finally, let h be a non-negative function such that $\int_{\gamma_{r_0}} h\geq 0$ for every $\gamma_{r_0}\in\Gamma_0$. Integrating with respect to r_0 we have

$$0\leq\int_r^R\frac{1}{t}\int_0^{2\pi}h(te^{i\theta})t\mathrm{d}\theta=\int_r^R\int_0^{2\pi}\rho(te^{i\theta})h(te^{i\theta})t\mathrm{d}\theta\mathrm{d}t=\iint_A h(z)\rho(z).$$

This proves that the metric is extremal. Finally we compute

$$A(A,\rho)=\iint_A\frac{1}{t^2}=\int_r^R\int_0^{2\pi}\frac{1}{t}\mathrm{d}\theta\mathrm{d}t=2\pi\log(R/r)$$

which together with $L(\Gamma,\rho)=2\pi$ gives

$$\lambda(\Gamma)=\frac{2\pi}{\log(R/r)}$$

For Γ' the extremal metric is the same and the proof that it is extremal is essentially the same (the family of the shortest curves is given by all radii). In this case

$$L=\int_r^R\frac{1}{t}\mathrm{d}t=\log(R/r)$$

and $A=2\pi\log(R/r)$ as before. Combining them together we have

$$\lambda(\Gamma')=\frac{1}{2\pi}\log(R/r)$$

Note: This is the conformal modulus that we discussed in the chapter about doubly connected domains.

Note: As expected $\lambda(\Gamma)\lambda(\Gamma')=1$. □

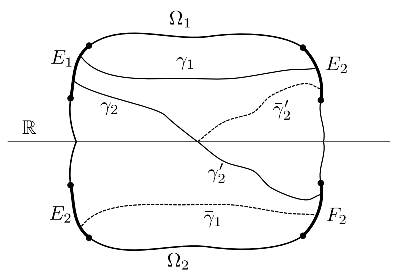
Problem 4.

Use the symmetry rule to prove the following statement.

Let Ω_1 be a domain in the upper half plane and let E_1 and F_1 be two sets on $\partial\Omega$. Let Ω_2, E_2, and F_2 be their symmetric images with respect to $\mathbb{R}$. We define $\Omega=\Omega_1\cup\Omega_2$ (to be completely rigorous we also have to add the real part of the boundary), $E=E_1\cup E_2$, and $F=F_1\cup F_2$. Then

$$2d_\Omega(E,F)=d_{\Omega_1}(E_1,F_1)=d_{\Omega_2}(E_2,F_2).$$

Solution. The domain Ω is symmetric and the family of the curves connecting E and F is symmetric, hence we can apply the symmetry rule and consider only symmetric metrics ρ.



For a curve γ connecting E and F we can consider its upper half-plane version γ' (real part of γ' is the same as the real part of γ and the imaginary part of γ' is the modulus of the imaginary part of γ). This curve is connecting E_1 and F_1

Since ρ is symmetric $L(\gamma, \rho) = L(\gamma, \rho)$ and $A(\Omega, \rho) = 2A(\Omega_1, \rho)$. Since every symmetric metric in Ω corresponds to a metric in Ω_1 we have that

$$d_\Omega(E, F) = \sup_\rho \frac{L^2(\Gamma, \rho)}{A(\Omega, \rho)} = \sup \frac{L^2(\Gamma_1, \rho)}{2A(\Omega_1, \rho)} = \frac{1}{2} d_{\Omega_1}(E_1, F_1)$$

□

Problem 5.

Let Ω be a simply connected domain, $z_0 \in \Omega$ and A be an arc (connected set) on the boundary of Ω. If you wish, you may assume that Ω is a nice domain, say, a domain bounded by an analytic Jordan curve, but this is not too important.

(1) Let Ω', z_0' and A' be another domain, a point and an arc as above. Show that there is a conformal map f such that $f(\Omega) = \Omega'$, $f(z_0) = z_0'$ and $f(A) = A'$ if and only if $\omega_\Omega(z_0, A) = \omega_{\Omega'}(z_0', A')$.

(2) Let Γ be the family of all rectifiable curves in Ω such that their endpoints are on A and they separate z_0 from $\partial\Omega \setminus A$. Show that there is a function F (independent of Ω, z_0 and A) such that $\lambda(\Gamma) = F(\omega_\Omega(z_0, A))$.

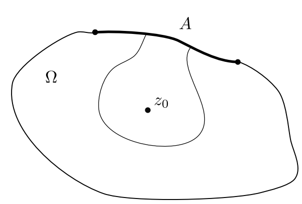


FIGURE 1. Family of curves Γ.

(3) By part (1) we can assume without loss of generality that $\Omega = \mathbb{D}$, $z_0 = 0$ and A is the arc $\{e^{i\theta}, -\theta_0 < \theta < \theta_0\}$ for some $\theta_0 \in [0, \pi)$. Let Γ be the family of curves as defined in part (2). Show that

$$\lambda(\Gamma) = 2d_{\mathbb{D}_+}([-1, 0], A_+) = 4\mathrm{d}_{\mathbb{D}\setminus[-1,0]}([-1, 0], A),$$

where $\mathbb{D}_+$ is the upper half-disc, A_+ is the upper half of A and $d_\Omega(E, F)$ is the extremal distance, that is the extremal length of the family of curves connecting boundary sets E and F inside Ω.

(4) Our next goal is to compute $d_{\mathbb{D}_+}([-1,0], A_+)$. We know that $\mathbb{D}_+$ with marked points $-1, 0, 1, e^{i\theta_0}$ could be mapped onto a rectangle in such a way that the marked points are mapped to the vertices. Use this fact to compute $d_{\mathbb{D}_+}([-1,0], A_+)$ in terms of θ_0. Combine all the results to find a formula for the function F from part (2).

(Hint: Use the fact that the upper half-plane with marked points $-1/k, -1, 1, 1/k$ could be mapped onto a rectangle with the ratio of side lengths equal to $2K(k)/K'(k)$, where K and K' are the complete elliptic integral of the the complementary complete elliptic integrals of the first king. You don't need to know anything about K and K', the only important thing is that they give an explicit expression for the side length ratio in terms of k.)

Solution. (1) Let ϕ be a conformal transformation from Ω onto $\mathbb{D}$ with $\phi(z_0) = 0$. This transformation sends A onto an arc of length

$$2\pi\omega_{\mathbb{D}}(0, \phi(A)) = 2\pi\omega_\Omega(z_0, A)$$

Two configurations are conformally equivalent if and only if their images in $\mathbb{D}$ are equivalent, which happens if and only if the corresponding arcs have the same length. By the formula above this means that harmonic measures must be equal.

(2) Extremal length is a conformal invariant of the configuration, which, by (1) is uniquely defined by the harmonic measure, this proves that $\lambda(\Gamma)$ depends only on $\omega(z_0, A)$

(3) By (1) we can conformally transform (Ω, z_0, A) onto $\mathbb{D}$, 0 and an arc $\{e^{i\theta}, -\theta_0 < \theta < \theta_0\}$ where $\theta_0 = \pi\omega_\Omega(z_0, A)$.

By the symmetry rule it is sufficient to consider symmetric metrics ρ. Next we want to symmetrize Γ

Let $\gamma(t) = (x(t), y(t))$ be any curve from Γ. Without loss of generality we can assume that its starting point is in the upper half-plane. Let us define $\gamma_+(t) = (x(t), |y(t)|)$ where $0 \le t \le t_+$ where t_+ is the first t such that $-1 < x(t) < 0$ and $y(t) = 0$, i.e. the first time γ crosses the interval $[-1, 0]$. After that we can join together γ_+ and its symmetric image to form a symmetric curve form Γ. In the similar way we can construct $\gamma_-(t) = (x(t), -|y(t)|)$ with $t_- < t$ where t_- is the last time the curve γ intersects $[-1, 0]$. As before we joint γ_- with its symmetric image. It is clear that for any symmetric metric at least one of these symmetric curves has the length bounded by the length of γ. This proves that in the definition of $L(\Gamma, \rho)$ we can consider only symmetric curves that do not cross $[0, 1]$ (but may touch it) and cross $[-1, 0]$ only once.

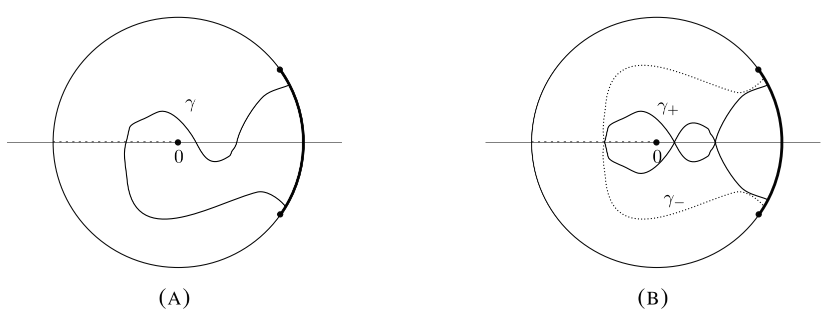


FIGURE 2. A curve γ (a) and its symmetrized versions γ_- and γ_+ (b).

By symmetry, the area of the disc is twice the area of the upper half-disc $\mathbb{D}_+$ and the length of a symmetric curve is twice the length of its upper half. This proves that $\lambda(\Gamma) = 2\lambda(\Gamma')$ where Γ' is the family of curves in the upper half-disc that connect the arc $A_+ = \{e^{i\theta}, 0 \le \theta \le \theta_0\}$ with the interval $F = [-1, 0]$. By the definition, $\lambda(\Gamma') = d_{\mathbb{D}_+}(A_+, F)$.

The same symmetry argument or problem 4 implies that

$$d_{\mathbb{D}_+}([-1,0], A_+) = 2\mathrm{d}_{\mathbb{D}\setminus[-1,0]}([-1,0], A)$$

(4) It is sufficient to compute the extremal distance between A_+ and $[-1, 1]$ in the upper half-disc. The function $J(z) = -z/2 - 1/2z$ conformally maps the upper half-disc onto $\mathbb{H}$ in such a way that $J([-1, 1]) = [-\infty, -1]$ and $J(E') = [\cos(\theta), 1]$. The simplest geometry where we can compute the extremal distance is the rectangle where it is the ratio of side lengths. To map it to the rectangle we first find a Möbius transformation that sends $(\infty, 0, \cos(\theta), 1)$ to $(-1/k, -1, 1, 1/k)$. Möbius transformations preserve cross-ratios, this means that such map exists if and only if cross-ratios are the same, this gives us a relation between $\cos(\theta)$ and k

$$\frac{2}{1+\cos(\theta)} = \frac{(1+k)^2}{4k}$$

or (since $0 < k < 1$)

$$k = k(\theta) = \frac{\sqrt{2} - \sqrt{1-\cos(\theta)}}{\sqrt{2} + \sqrt{1-\cos(\theta)}}.$$

Christoffel-Schwarz function maps $\mathbb{H}$ with these marked points onto a rectangle with ratio of side lengths

$$\frac{2K(k)}{K'(k)}.$$

Hence $\lambda(\Gamma) = 4K(k)/K'(k)$ where K and K' are elliptic integrals and $k = k(\theta)$. □

Problem 6.

Let Ω be a conformal triangle, i.e. a simply connected domain bounded by a Jordan curve with three marked points on it. We will call these marked points the vertices and the arcs between them the sides of the conformal triangle Ω. Let γ be a continuous curve in $\bar{\Omega}$ such that it intersects with all three sides of Ω.

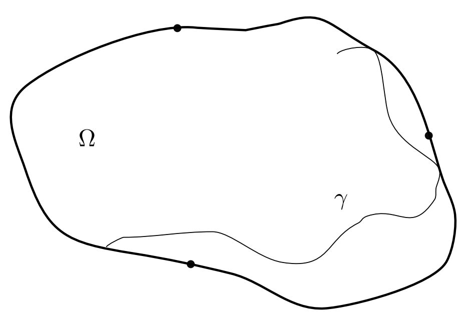


FIGURE 3. Conformal triangle and a curve inside touching all three sides.

Use extremal lengths to show that there exists a curve γ as above such that

$$L(\gamma) \leq 3^{1/4}\sqrt{A(\Omega)}$$

where $L(\gamma)$ is the usual Euclidean length of γ and $A(\Omega)$ is the area of Ω.

Show that the constant $3^{1/4}$ is sharp.

Solution. First of all we have to formulate the question in terms of extremal length. Let Γ be the family of curves connecting all three sides. Since all conformal triangles are conformally equivalent, $\lambda(\Gamma)$ is just an absolute constant independent of everything.

Let us consider $\rho = 0$. By the definition

$$\lambda(\Gamma) \geq \inf_{\gamma} \frac{L^2(\gamma)}{A(\Omega)}$$

which implies that there is γ such that $L(\gamma) \leq \sqrt{\lambda(\Gamma)A(\Omega)}$. We just have to compute $\lambda(\Gamma)$. By conformal invariance it is enough to consider the case of equilateral triangle with side length 1.

Let Γ_0 be the union of three families of curves. In each the curves start as straight lines orthogonal to one of the sides, reflect of the other side and end on the third side. It is easy to see that all these curves have the same length $\sqrt{3}/2$ with respect to $\rho = 1$. It is also easy to check that this ρ is indeed extremal. This proves that $\lambda(\Gamma) = (3/4)/(\sqrt{3}/4) = \sqrt{3}$.

$\square$