C4.8 Complex Analysis: conformal maps and geometry

Sheet 0

Solutions

Problem 1.

Let γ be any simple closed curve inside Ω which encompasses z_0. By the Cauchy formula for f_n' we have

$$f_n'(z_0) = \frac{1}{2\pi} \int_\gamma \frac{f_n(\zeta)}{(\zeta - z_0)^2} \mathrm{d}\zeta.$$

Since γ is a compact set, $f_n(\zeta)/(\zeta - z_0)^2 \to f(\zeta)/(\zeta - z_0)^2$ uniformly on γ. This allows to pass to the limit under the integral sign and to show that

$$f_n'(z_0) \to \frac{1}{2\pi} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^2} \mathrm{d}\zeta = f'(z_0).$$

Problem 2.

(1) This is probably the most standard proof. Let $p(z) = a_n z^n + \cdots + a_0$ be a polynomial. It is easy to see that there is $R > 0$ such that $|p(z)| > |a_n||z^n|/2$ for $|z| > R$. Let us assume that p has no roots. In this case $f(z) = 1/p(z)$ is an entire function. Inside the disc $|z| \leq R$ function f is continuous, hence bounded. Outside, it is bounded by $2/(|a_n|R^n)$. Hence, by Liouville theorem, f must be constant. This implies that p is also constant, which contradicts our assumption that the degree of p is at least 1.

(2) This proof is quite similar to the one above. Let us consider $f = 1/p$. If p has no roots that f is analytic in any disc. By the maximum modulus theorem

$$\sup_{|z| \leq R} |f(z)| \leq \sup_{|z| = R} |f(z)|.$$

By the same argument as above, the right hand side goes to zero as $R \to \infty$. This proves that $f = 0$ everywhere which is impossible.

(3) Let us consider

$$\frac{1}{2\pi i} \int_{|z|=R} \frac{p'(z)}{p(z)} \mathrm{d}z.$$

By argument principle this integral gives the number of roots inside $|z| = R$. By the standard argument, when R is sufficiently large, there are no roots outside, hence, this integral gives the total number of roots. On the other hand it is equal to the winding number of the image of $\gamma = \{z : |z| = R\}$. But for large R the image of γ is close to the image under $a_n z^n$ which goes n times around z. This proves that p has exactly n roots.

(4) Let us consider $f(z) = a_n z^n$. Arguing as above $|p(z) - f(z)| < |f(z)|$ when $|z|$ is sufficiently large. By Rouché theorem this implies that the numbers of roots of f and p inside the disc of sufficiently large radius are the same. Since f has exactly n roots, so does p.

Problem 3.

A $\mathbb{C}$-valued function has a removable singularity at infinity if and only if there is finite limit of $f(z)$ as $z \to \infty$. This is equivalent to $g(z) = f(1/z)$ having a removable singularity at 0. It is a standard fact that g has a removable singularity if it has no singular part in its Laurent expansion. This is the same $a_n = 0$ for all $n \geq 1$.

When we consider $\widehat{\mathbb{C}}$-valued functions, the situation is quite different. In this case it is acceptable for g to converge to infinity. It is a standard fact about isolated singularities, that $g(z) \to \infty$ as $z \to 0$ if and only if 0 is a pole of g. This is equivalent to the statement that there are only finitely many non-zero a_n with $n \geq 1$.

Finally, let us assume that f is one-to-one near infinity. By the argument above, $f \to \infty$ as $z \to \infty$ means that g has a pole at 0 (and also one-to-one). This is equivalent to $h(z) = 1/f(1/z)$ being one-to-one near the origin. Unless f is of the form

$$f(z) = a_1 z + \sum_{-\infty}^{0} a_n z^n,$$

the function h has a zero of order higher than one. In this case $h'(0) = 0$ and h is not locally one-to-one. This contradicts our assumption that f is one-to-one.

Problem 4.

Let μ be the Möbius transformation which maps z_1, z_2 and z_3 to ∞, 0, and 1 correspondingly. In this case $\mu(z_4)$ is equal to the cross-ratio. This proves that $\mu(z)$ is real if and only if the cross-ratio is real. On the other hand, $\mu(z_4)$ is real if and only if images of all points are on the real line, that is they a collinear. All four points z_i lie on the preimage of the real line, which is either a line or a circle.