C4.8 Complex Analysis: conformal maps and geometry

Sheet 1

Problem 1.

Show that the only Möbius transformations that map $\mathbb{D}$, $\mathbb{C}$ or $\mathbb{H}$ to themselves are of the form

$$
\begin{aligned}
e^{i\theta}\frac{z-a}{1-\bar{a}z}, &\quad a\in\mathbb{D},\ \theta\in\mathbb{R}\\
az+b, &\quad a,b\in\mathbb{C}\\
\frac{az+b}{cz+d}, &\quad a,b,c,d\in\mathbb{R},\ ad-bc>0.
\end{aligned}
$$

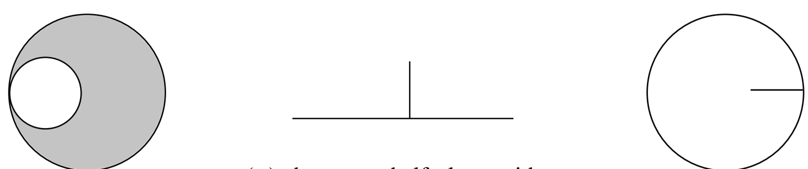
Problem 2.

Prove that all conformal automorphisms of $\widehat{\mathbb{C}}$, $\mathbb{C}$, and $\mathbb{H}$ are Möbius transformations.

Problem 3.

For the following domains find a conformal map onto $\mathbb{D}$ or $\mathbb{H}$.

(1) Infinite strip, $0 < \Im z < 1$.
(2) Domain bounded by two touching circles. Namely, let Let Ω be the domain between two circles or radii r and R centred at r and R. See Figure 1a.
(3) The upper half-plane with a slit $\Omega = \mathbb{H}\setminus[0, it]$ with $t > 0$, see Figure 1b. *(Hint: You might start with applying $z \mapsto z^2$.)*
(4) The unit disc with a slit $\mathbb{D}\setminus[x, 1]$ with $-1 < x < 1$, see Figure 1c.



(A) Two circles   (B) the upper half-plane with a slit   (C) A disc with a slit

FIGURE 1. Three domains where uniformizing maps could be found explicitly.

Problem 4.

Let $\mathcal{F}$ be the family of all functions of the form $f_w(z) = z/(z-w)$ with $|w| > 1$. Show without use of Montel's theorem that this family is normal in the unit disc.

Problem 5.

Show that the family $\mathcal{F}$ of all holomorphic functions in $\mathbb{D}$ with positive real part is normal.

Problem 6.

We define the Hardy space $H^1 = H^1(\mathbb{D})$ as the space of all holomorphic functions in $\mathbb{D}$ such that

$$\|f\|_{H^1} = \sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta}|\mathrm{d}\theta < \infty.$$

Show that the family of functions $f \in H^1$ such that $\|f\|_{H^1} \le 1$ is normal.