C4.8 Complex Analysis: conformal maps and geometry

Sheet 2

Problem 1.

Find explicit formulas (that might involve special functions) for conformal maps between Ω and one of the standard uniformizing domains where Ω is

(1) Semi-infinite strip $S = \{z : -\pi/2 < \Re z < \pi/2,\ \Im z > 0\}$.
(2) Equilateral triangle.

In both cases you can find either a map from Ω onto a uniformizing domain or from a uniformizing domain onto Ω. Choose one which is easier to find. *(The answer might involve functions in the integral form. All these functions have special names, in particular, in the second part you will encounter Beta function, Gamma function, and a hypergeometric function.)*

Problem 2.

Let R and R' be two rectangles and let $\lambda, \lambda' \geq 1$ be the ratios of the side lengths of R and R'. Assume that there is a conformal map $f : R \to R'$ which is continuous up to the boundary and maps vertices to vertices. Show that $\lambda = \lambda'$.

(Hint: Do not try to write the map f explicitly.)

Problem 3. You probably have seen Schwarz reflection principle before. Let Ω be a domain in $\mathbb{H}$ and let $I = \partial\Omega \cap \mathbb{R}$, let f be a function analytic in Ω and continuous in $\Omega \cup I$ such that the values on I are real. If we define $f(z) = \overline{f(\overline{z})}$ in the domain $\bar{\Omega}$ (symmetric image of Ω w.r.t. the real line), then f is analytic in the interior of the union of Ω, $\bar{\Omega}$, and I.

Similar statement holds when we use the symmetry with respect to a circle instead of the real line. Let f be an analytic function in an annulus $A(r, R) = \{z : r < |z| < R\}$ which is continuous up to the boundary and such that $|f(z)| = R'$ for $|z| = R$. Then we can extend the function f to $A(R, R^2/r)$ by $f(z) = (R')^2/\overline{f(R^2/\overline{z})}$. Then the function f is analytic in $A(r, R^2/r)$.

Use the reflection principle to prove the following statement: Let A_1 and A_2 be two non-degenerated annuli. Let us also assume that there is a conformal map $f : A_1 \to A_2$ which is continuous up to the boundary. Then A_1 and A_2 have the same shape.

Problem 4.

Let S_1 and S_2 be two circles such that S_1 is inside S_2 (but not necessarily concentric) and let Ω be a domain between these two circles. Find an explicit conformal map from Ω onto an annulus.

Problem 5.

From the lectures we know that any n-connected domain Ω could be mapped on a parallel slit domain. Moreover, for any fixed $z_0 \in \Omega$ there is a unique univalent map $f_{z_0,\theta}$ from Ω onto a parallel slit domain such that all slits form the angle θ with the real line, z_0 is mapped to infinity, and $f_{z_0,\theta}$ has Laurent series at z_0 of the form

$$\frac{1}{z - z_0} + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots$$

Let us assume that $n > 2$. Use the fact that Ω could be mapped to a parallel slit domain to explain why the conformal type of Ω could be completely determined by $3n - 6$ real parameters. This means that to each domain we can prescribe $3n - 6$ parameters and the domains are conformally equivalent to each over if and only if the parameters are the same. *(Note: these parameters are called Riemann moduli.)*

Problem 6. In this problem we assume that Ω and Ω' are two simply connected domain and $f : \Omega \to \Omega'$ a conformal transformation between them. For simplicity we can also assume that the domains are bounded and that f extends to a continuous bijection between boundaries of Ω and Ω'.

(1) Let $h : \Omega' \to \mathbb{R}$ be a harmonic function in Ω'. Show that $g(z) = h(f(z))$ is a harmonic function in Ω.

(2) Let us recall that the Green's function with pole at $z_0 \in \Omega$ is defined as the unique function such that $G_\Omega(z_0, z) = G(z_0, z)$ is harmonic for $z \in \Omega \backslash \{z_0\}$, $G(z_0, z) \to 0$ if $z \to \partial\Omega$, and $G(z_0, z) + \log|z - z_0| = u(z)$ is a bounded harmonic function in Ω.

Show that $G_{\Omega'}(f(z_0), f(z)) = G_\Omega(z_0, z)$.

(Hint: It is useful to remember that the real and imaginary parts of an analytic function are harmonic and that the real part of logarithm is always single-valued.)

(3) Check that $G_{\mathbb{D}}(0, z) = -\log|z|$ and

$$G_{\mathbb{D}}(z, w) = \log\left|\frac{1 - z\bar{w}}{z - w}\right|.$$

(4) Let ϕ be a conformal map from Ω onto $\mathbb{D}$ such that $\phi(z_0) = 0$. Show that $G_\Omega(z_0, z) = -\log|\phi(z)|$.

(5) (Bonus question). In the previous part we have shown that we can find the Green's function if we know the Riemann's map. Can we find the Riemann map ϕ if we know G_Ω?