C4.3 Functional Analytic Methods for PDEs: Q2(a)(ii) – 2018

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Given:

- $\Omega \subset \mathbb{R}^n$: bounded C^1 domain.
- $1 \leq p < \infty$.

Want: Prove the Poincaré-Sobolev inequality

$$\|u - [u]_{\partial\Omega}\|_{L^p(\Omega)} \leq C(\Omega, p, n)\|Du\|_{L^p(\Omega)} \text{ for all } u \in W^{1,p}(\Omega).$$

Here

$$[u]_{\partial\Omega} = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} u \, \mathrm{d}x.$$

Start of proof

- Suppose by contradiction that the conclusion fails. Then there exist $0 \not\equiv u_j \in W^{1,p}(\Omega)$ such that

$$\|u_j - [u_j]_{\partial\Omega}\|_{L^p(\Omega)} \geq j\|Du_j\|_{L^p(\Omega)}.$$

- Replacing u_j by $u_j - [u_j]_{\partial\Omega}$, we may assume that $[u_j]_{\partial\Omega} = 0$. Then

$$\|u_j\|_{L^p(\Omega)} \geq j\|Du_j\|_{L^p(\Omega)}.$$

- Replacing u_j by $\frac{1}{\|u_j\|_{L^p(\Omega)}} u_j$, we may assume that $\|u_j\|_{L^p(\Omega)} = 1$. Then

$$1 = \|u_j\|_{L^p(\Omega)} \geq j\|Du_j\|_{L^p(\Omega)}.$$

- In particular (u_j) is bounded in $W^{1,p}(\Omega)$ and (Du_j) converges strongly to 0 in $L^p(\Omega)$.

Use of Rellich-Kondrachov's theorem

- By Rellich-Kondrachov's theorem, the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact. Thus, passing to a subsequence if necessary, we may assume that (u_j) converges strongly in $L^p(\Omega)$ to some $u \in L^p(\Omega)$.
- As $Du_j \to 0$ in $L^p(\Omega)$, we have $Du = 0$. As Ω is a connected (since it is a domain), we have $u \equiv a$ for some constant a.
- We then have $u_j \to a$ in $W^{1,p}(\Omega)$. One one hand this implies

$$1 = \|u_j\|_{L^p(\Omega)} \to |a||\Omega|^{1/p} \text{ and so } a \neq 0.$$

  On the other hand, by the trace theorem, $u_j|_{\partial\Omega}$ converges to a in $L^p(\partial\Omega)$, hence in $L^1(\Omega)$ and so

$$0 = [u_j]_{\partial\Omega} \to a,$$

  which amounts to a contradiction.