C4.3 Functional Analytic Methods for PDEs: Q2(b) – 2018

Luc Nguyen

University of Oxford

May 2020

Q2(b) – 2018

Given:

- $\Omega \subset \mathbb{R}^n$: bounded Lipschitz domain.
- $u \in H^2(\Omega)$.

Want:

(ii)
- If $n = 3$, then u is Hölder continuous in $\bar{\Omega}$.
- Is the above true for $n = 4$?

$n = 3$: Sobolev + Morrey's embedding

- We have $u, Du \in W^{1,2}(\Omega)$ and so $u, Du \in L^{2^*}(\Omega) = L^6(\Omega)$.
- Hence $u \in W^{1,6}(\Omega)$ and so $u \in C^{0,\frac{1}{2}}(\bar{\Omega})$.

$n = 4$: Limiting case of Sobolev embedding

- We now have $u, Du \in L^{2^*}(\Omega) = L^4(\Omega)$, i.e. $u \in W^{1,4}(\Omega)$. This is the threadhold case in dimension 4.
- Claim u need not be continuous. Consider $u(x) = |\ln|x||^\alpha$ for some $\alpha > 0$ in $\Omega = B_R$ for some $R < 1$. Clearly u is unbounded in Ω. If we manage to get α such that $u \in H^2(\Omega)$, we are done. We compute

$$\int_\Omega |u|^2\,dx = \int_0^R r^3|\ln r|^\alpha\,dr < \infty.$$

$n = 4$: L^2 norm of u and Du

- $$\int_\Omega |u|^2\,dx = \int_0^R r^3 |\ln r|^{2\alpha}\,dr < \infty.$$

- $$\int_\Omega |Du|^2\,dx = \int_0^R |\partial_r u|^2\, r^3\,dr = \alpha^2 \int_0^R r |\ln r|^{2\alpha-2}\,dr < \infty.$$

$n = 4$: L^2 norm of $D^2 u$

- $$\begin{aligned}\int_\Omega |D^2 u|^2\,dx &= \int_0^R \Big(|\partial_r^2 u|^2 + \frac{3}{r^2}|\partial_r u|^2\Big)\, r^3\,dr\\ &= \alpha^2 \int_0^R \frac{1}{r}\Big[|\ln r|^{\alpha-1} + (\alpha-1)|\ln r|^{\alpha-2}\Big]^2\,dr\\ &\quad + 3\alpha^2 \int_0^R \frac{1}{r}|\ln r|^{2\alpha-2}\,dr\\ &\le C(\alpha)\int_0^R \frac{1}{r}|\ln r|^{2\alpha-2}\,dr.\end{aligned}$$

This is bounded provided $2\alpha - 2 < -1$, i.e. $\alpha < 1/2$.