C4.3 Functional Analytic Methods for PDEs: Q1(b)(c) – 2019

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Q1(b)(c) – 2019

Given:

- $\Omega \subset \mathbb{R}^n$: bounded Lipschitz domain.
- $\overline{W}^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : \int_\Omega u\,dx = 0\}$.

Want:

(b) For $1 \le p < n$,

$$\|u\|_{L^{p^*}(\Omega)} \le C\|Du\|_{L^p(\Omega)} \text{ for all } u \in \overline{W}^{1,p}(\Omega).$$

(c) (i) For $n = 3$ and $2 < q < 6$,

$$\|u\|_{L^q(\Omega)} \le C'\|u\|_{L^2(\Omega)}^{\frac{6-q}{2q}}\|Du\|_{L^2(\Omega)}^{\frac{3(q-2)}{2q}} \text{ for all } u \in \overline{W}^{1,2}(\Omega).$$

TYPO as $\|u\|_{L^2(\Omega)}^{\frac{3(q-2)}{2q}}\|Du\|_{L^2(\Omega)}^{\frac{6-q}{2q}}$

(ii) Deduce that, for any $\varepsilon > 0$,

$$\|u\|_{L^q(\Omega)} \le \varepsilon\|Du\|_{L^2(\Omega)} + C''_\varepsilon\|u\|_{L^2(\Omega)} \text{ for all } u \in W^{1,2}(\Omega).$$

Part (b): Proof by extension

- As in the course, we reduce to the whole space setting. There is a bounded linear extension $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$.
- By Gargliardo-Nirenberg's inequality, we have for $u \in W^{1,p}(\Omega)$ that

$$\|u\|_{L^{p^*}(\Omega)} \leq \|Eu\|_{L^{p^*}(\mathbb{R}^n)} \leq C_{n,p}\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|E\|\|u\|_{W^{1,p}(\Omega)}$$

- By Poincaré-Sobolev's inequality, we have for $u \in \overline{W}^{1,p}(\Omega)$ that

$$\|u\|_{L^p(\Omega)} \leq C\|Du\|_{L^p(\Omega)}$$

  and so

$$\|u\|_{W^{1,p}(\Omega)} \leq C\|Du\|_{L^p(\Omega)}$$

- Chaining together we are done.

Part (c)(i): Interpolation

- Interpolation inequality: $\|u\|_{L^q(\Omega)} \leq \|u\|_{L^r(\Omega)}^{\theta} \|u\|_{L^s(\Omega)}^{1-\theta}$ if $q \in [r, s]$ and θ is such that $\frac{1}{q} = \frac{\theta}{r} + \frac{1-\theta}{s}$.
- This is a consequence of Hölder's inequality. We start by writing

$$\|u\|_{L^q(\Omega)}^q = \int_\Omega |u|^q \, dx = \int_\Omega u^{q\theta} \, u^{q(1-\theta)} \, dx.$$

- Observe that $\frac{1}{q} = \frac{\theta}{r} + \frac{1-\theta}{s}$ implies $1 = \frac{q\theta}{r} + \frac{q(1-\theta)}{s}$. We can thus use Hölder's inequality with exponent $\frac{r}{q\theta}$ and $\frac{s}{q(1-\theta)}$ to obtain

$$\|u\|_{L^q(\Omega)}^q \leq \Big(\int_\Omega u^r \, dx \Big)^{\frac{q\theta}{r}} \Big(\int_\Omega u^s \, dx \Big)^{\frac{q(1-\theta)}{s}} = \|u\|_{L^r(\Omega)}^{q\theta} \|u\|_{L^s(\Omega)}^{q(1-\theta)}.$$

Part (c)(i): Conclusion

- Recall that we are dealing with $n = 3$ and $\overline{W}^{1,2}(\Omega)$, i.e. $p = 2$. So $p^* = 6$. By (b), we have

$$\|u\|_{L^6(\Omega)} \le C\|Du\|_{L^2(\Omega)} \text{ for all } u \in \overline{W}^{1,2}(\Omega).$$

- By interpolation inequality, we have $\|u\|_{L^q(\Omega)} \le \|u\|_{L^2(\Omega)}^{\theta}\|u\|_{L^6(\Omega)}^{1-\theta}$ where $\frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{6}$, i.e. $\theta = \frac{6-q}{2q}$.
- Putting together we get

$$\|u\|_{L^q(\Omega)} \le C\|u\|_{L^2(\Omega)}^{\frac{6-q}{2q}}\|Du\|_{L^2(\Omega)}^{\frac{3(q-2)}{2q}} \text{ for all } u \in \overline{W}^{1,2}(\Omega).$$

Part (c)(ii): Young's inequality

- Young's inequality: $ab \le \frac{1}{p}a^p + \frac{1}{p'}b^{p'}$ for $1 < p < \infty$.
- Suppose that $u \in W^{1,2}(\Omega)$. Let $[u]_\Omega = \frac{1}{|\Omega|}\int_\Omega u\,dx$ be the average of u and $\bar{u} = u - [u]_\Omega$ so that $\bar{u} \in \overline{W}^{1,2}(\Omega)$.
- By (c)(i),

$$\|\bar{u}\|_{L^q(\Omega)} \le C\|\bar{u}\|_{L^2(\Omega)}^{\frac{6-q}{2q}}\|Du\|_{L^2(\Omega)}^{\frac{3(q-2)}{2q}}.$$

- Applying Young's inequality with $p = \frac{2q}{3(q-2)}$, we obtain

$$\|\bar{u}\|_{L^q(\Omega)} \le \varepsilon\|Du\|_{L^2(\Omega)} + C_\varepsilon\|\bar{u}\|_{L^2(\Omega)}.$$

- By triangle inequality,

$$\|u\|_{L^q(\Omega)} \le \varepsilon\|Du\|_{L^2(\Omega)} + C_\varepsilon\|\bar{u}\|_{L^2(\Omega)} + C\underbrace{|u_\Omega|}_{\le C\|u\|_{L^2(\Omega)}}$$

$$\le \varepsilon\|Du\|_{L^2(\Omega)} + C_\varepsilon\|u\|_{L^2(\Omega)}.$$