C4.3 Functional Analytic Methods for PDEs - Sheet 1 of 4

Read Chapter 1 and prove the few statements whose proofs were left out as exercises. (Not to be handed in.)

Do:

Q1. Let $E \subset \mathbb{R}^n$ be measurable and $f_j, f : E \to \mathbb{R}$ be measurable.

(i) Prove that if $f_j \to f$ a.e. in E and if E has finite measure, then $f_j \to f$ in measure in E.

(ii) Prove that if $f_j \to f$ in measure in E, then there is a subsequence f_{j_k} such that $f_{j_k} \to f$ a.e. in E.

(iii) Prove that if $f_j \to f$ in $L^p(E)$ for some $1 \le p \le \infty$, then $f_j \to f$ in measure.

Q2. For what $1 \le p \le \infty$ and measurable $E \subset \mathbb{R}$, can $L^p(E)$ with its standard norm be made a Hilbert space?

Q3. Prove Young's convolution inequality $\|f * g\|_{L^r(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}$ when $1 \le p, q, r \le \infty$ satisfy $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. [*For $f, g \ge 0$ and $p, q, r < \infty$, write*

$$(f * g)(x) = \int_{\mathbb{R}^n} [f(y)^{\frac{p}{r}} g(x-y)^{\frac{q}{r}}][f(y)^{1-\frac{p}{r}}][g(x-y)^{1-\frac{q}{r}}]\, dy$$

and apply Hölder's inequality for three functions with suitable exponents.]

Q4. (i) Let $E \subset \mathbb{R}^n$ be a measurable set of finite measure. Show that, for every $\lambda > 0$, the set $E_\lambda := \{\lambda x : x \in E\}$ is measurable and $|E_\lambda| = \lambda^n |E|$. [*You may want to consider first the cases E is a cube, an open set or a compact set, before considering the general case.*]

(ii) Let $h \in L^1(\mathbb{R}^n)$. By approximating h by simple functions, or otherwise, show that, for every $\lambda > 0$,

$$\int_{\mathbb{R}^n} h(\lambda x)\, dx = \frac{1}{\lambda^n} \int_{\mathbb{R}^n} h(x)\, dx.$$

(iii) Let $f \in L^p(\mathbb{R}^n)$ for some $1 \le p < \infty$. For $\lambda > 0$, define $f_\lambda(x) = f(\lambda x)$. Show that $f_\lambda \in L^p(\mathbb{R}^n)$ for every $\lambda > 0$ and

$$\lim_{\lambda \to 1} \|f_\lambda - f\|_{L^p(\mathbb{R}^n)} = 0.$$

Q5. (i) By considering the family $\{\chi_{(0,t)}\}_{t\in(0,1)}$, or otherwise, show that $L^\infty(0,1)$ is not separable.

(ii) Show that $L^1(0,1)$ is a proper subspace of $(L^\infty(0,1))^*$.

Q6. Let Ω be a bounded domain in $\mathbb{R}^n$.

(i) Show that, for every $1 \le p < \infty$, $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$.

(ii) Is $C_c^\infty(\Omega)$ dense in $L^\infty(\Omega)$?