C4.3 Functional Analytic Methods for PDEs - Sheet 2 of 4

Read Chapter 2 and prove the few statements whose proofs were left out as exercises. (Not to be handed in.)

Do:

Q1. Let $f_\alpha(x) = x^\alpha$ for $x \in (0,1)$ and $\alpha \in \mathbb{R}$. Compute its weak derivative f'_α and determine, for given $1 \le p < \infty$ the values of α such that $f_\alpha \in W^{1,p}(0,1)$.

Q2. Let Ω be a domain in $\mathbb{R}^n$ and $u \in L^p_{loc}(\Omega)$, $1 \le p < \infty$. Let $\varrho \in C_c^\infty(B_1)$ be a non-negative function satisfying $\int_{\mathbb{R}^n} \varrho = 1$. For $\varepsilon > 0$, let $\Omega_\varepsilon := \{x \in \Omega : B_\varepsilon(x) \subset \Omega\}$ and $\varrho_\varepsilon(x) = \varepsilon^{-n}\varrho(x/\varepsilon)$.

(i) Show that $u_\varepsilon := \varrho_\varepsilon * u$ is well-defined in Ω_ε and u_ε converges to u in $L^p_{loc}(\Omega)$ (i.e. $u_\varepsilon \to u$ in $L^p(K)$ for every compact subset K of Ω).

(ii) Show that if u has first order weak derivative Du, then $Du_\varepsilon = \varrho_\varepsilon * Du$ in Ω_ε.

(iii) Deduce that if $Du = 0$ a.e. in Ω, then $u = C$ a.e. in Ω for some constant C.

Q3. (Chain rule) Let Ω be a bounded domain in $\mathbb{R}^n$ and assume that $u \in W^{1,p}(\Omega)$ with $1 \le p < \infty$. Let $f : \mathbb{R} \to \mathbb{R}$ be C^1 and satisfy $|f'| \le L < \infty$ in $\mathbb{R}$. In this exercise, we are going to show that $w = f(u)$ belongs to $W^{1,p}(\Omega)$ and $Dw = f'(u)Du$.

(i) Using the inequality $|f(t)| \le |f(0)| + L|t|$, show that $w \in L^p(\Omega)$.

Let $\varrho \in C_c^\infty(B_1)$ be a non-negative function satisfying $\int_{\mathbb{R}^n} \varrho = 1$ and ϱ_ε be defined by $\varrho_\varepsilon(x) = \varepsilon^{-n}\varrho(x/\varepsilon)$, $\varepsilon > 0$. Define

$$\Omega_\varepsilon = \{x \in \Omega : B_\varepsilon(x) \subset \Omega\},$$
$$u_\varepsilon = \varrho_\varepsilon * u \text{ and } w_\varepsilon = f(u_\varepsilon) \text{ in } \Omega_\varepsilon.$$

(ii) Show that if K is a compact subset of Ω, then

$$\|w_\varepsilon - w\|_{L^p(K)} \le C\|u_\varepsilon - u\|_{L^p(K)}$$

and deduce that $w_\varepsilon \to w$ in $L^p_{loc}(\Omega)$.

(iii) Show that $f'(u_\varepsilon)Du$ converges along a subsequence to $f'(u)Du$ in $L^p_{loc}(\Omega)$.

(iv) Show that Dw_ε converges along a subsequence to $f'(u)Du$ in $L^p_{loc}(\Omega)$. Deduce that w has a first order weak derivative $Dw = f'(u)Du$. Conclude that $w \in W^{1,p}(\Omega)$.

Q4. (A special case of Theorem 2.3.5) Let $1 \le p < \infty$, Ω be a domain in $\mathbb{R}^n$ and suppose that Ω is star-shaped in the sense that there exists a point $x_0 \in \Omega$ such that for every $x \in \Omega$, the line segment $[xx_0]$ connecting x and x_0 stays in Ω. Let $u \in W^{1,p}(\Omega)$. For $\lambda > 0$, let $\Omega^\lambda = \{x : x/\lambda \in \Omega\}$ and $u^\lambda(x) = u(x/\lambda)$ for $x \in \Omega^\lambda$. Show that $u^\lambda \in W^{1,p}(\Omega^\lambda)$. Applying suitable mollification to u^λ with λ close to 1, show that u can be approximated by functions in $C^\infty(\bar{\Omega})$. Deduce that $C^\infty(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$.

Q5. (Local extension) Let $1 \le p < \infty$, B be the open unit ball, $B^+ = \{(x', x_n) \in B, x_n > 0\}$ and $B^- = \{(x', x_n) \in B, x_n < 0\}$.

(i) Let $u \in C^1(\bar{B}^+)$ and define, for $x \in \bar{B}$,

$$\bar{u}(x) = \begin{cases} u(x) & \text{if } u \in \bar{B}^+, \\ -3u(x', -x_n) + 4u(x', -\frac{1}{2}x_n) & \text{if } u \in \bar{B}^-. \end{cases}$$

Show that $\bar{u} \in C^1(\bar{B})$ and $\|\bar{u}\|_{W^{1,p}(B)} \le C_* \|u\|_{W^{1,p}(B^+)}$, where the constant C_* is independent of u.

(ii) Deduce that there is an extension operator $E : W^{1,p}(B^+) \to W^{1,p}(B)$ such that $Eu = u$ a.e. in B^+ and

$$\|Eu\|_{W^{1,p}(B)} \le C_* \|u\|_{W^{1,p}(B^+)} \text{ for all } u \in W^{1,p}(B^+).$$

Q6. Suppose $1 < p < \infty$ and $p' = \frac{p}{p-1}$.

(i) (Integration by parts) Let Ω be a bounded domain with Lipschitz boundary. Let ν be the outward pointing unit normal to $\partial\Omega$. By mean of approximations, show that for every $i \in \{1, \dots, n\}$, $u \in W^{1,p}(\Omega)$ and $v \in W^{1,p'}(\Omega)$ there holds

$$\int_\Omega D_i u v = \int_{\partial\Omega} u v \nu_i - \int_\Omega u D_i v.$$

Here the values of u and v on $\partial\Omega$ are understood in the sense of trace.

(ii) Suppose $u^- \in W^{1,p}(B_1)$ and $u^+ \in W^{1,p}(B_2 \setminus B_1)$, where B_1 and B_2 are concentric balls of radii 1 and 2, respectively. Suppose that the trace of u^- on ∂B_1 coincides with the (restriction of the) trace of u^+ on ∂B_1. Show that the function u defined by

$$u(x) = \begin{cases} u^-(x) & \text{if } x \in B_1, \\ u^+(x) & \text{if } x \in B_2 \setminus B_1, \end{cases}$$

belongs to $W^{1,p}(B_2)$.