C4.3 Functional Analytic Methods for PDEs - Sheet 2 of 4

Q4. (A special case of Theorem 2.3.5) Let $1 \le p < \infty$, Ω be a domain in $\mathbb{R}^n$ and suppose that Ω is star-shaped in the sense that there exists a point $x_0 \in \Omega$ such that for every $x \in \Omega$, the line segment $[xx_0]$ connecting x and x_0 stays in Ω. Let $u \in W^{1,p}(\Omega)$. For $\lambda > 0$, let $\Omega^\lambda = \{x : x/\lambda \in \Omega\}$ and $u^\lambda(x) = u(x/\lambda)$ for $x \in \Omega^\lambda$. Show that $u^\lambda \in W^{1,p}(\Omega^\lambda)$. Applying suitable mollification to u^λ with λ close to 1, show that u can be approximated by functions in $C^\infty(\bar{\Omega})$. Deduce that $C^\infty(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$.

Sketched solution. We may assume that Ω is star-shaped about the origin.

Let $u \in W^{1,p}(\Omega)$. Note that, for $\underline{\underline{\lambda > 1}}$, Ω^λ contains $\bar{\Omega}$ and, by a problem in Sheet 1 (applied to u and $\partial_i u$; please fill in the details), $u^\lambda \in W^{1,p}(\Omega^\lambda)$ and

$$\|u^\lambda - u\|_{W^{1,p}(\Omega)} \to 0 \text{ as } \lambda \to 1.$$

Now, for all sufficiently small ε, the mollification $u^\lambda * \varrho_\varepsilon$ belongs to $C^\infty(\Omega^\lambda)$ and

$$\|u^\lambda * \varrho_\varepsilon - u^\lambda\|_{W^{1,p}(\Omega)} \to 0 \text{ as } \varepsilon \to 0.$$

The approximation sequence for u is then constructed as follows: For $k > 0$, select first a $\lambda_k > 1$ such that $\|u^{\lambda_k} - u\|_{W^{1,p}(\Omega)} < 1/k$. Then select ε_k sufficiently small such that $u^{\lambda_k} * \varrho_{\varepsilon_k}$ is defined on $\bar{\Omega}$ and $\|u^{\lambda_k} * \varrho_{\varepsilon_k} - u^{\lambda_k}\|_{W^{1,p}(\Omega)} < 1/k$. Let $u_k = u^{\lambda_k} * \varrho_{\varepsilon_k} \in C^\infty(\bar{\Omega})$ we then see that $\|u_k - u\|_{W^{1,p}(\Omega)} < 2/k$, which gives the result. $\square$