Functional Analytic Methods for PDE's

G. Seregin

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Chapter 1

Introduction

1.1 Why Functional Analysis Methods are important for PDE's?

Let us consider an important example: a linear PDE (partial differential equation) of elliptic type

$$Lu := -(a_{ij}u_{,j})_{,i} + b_i u_{,i} + cu = f + g_{i,i} \qquad \text{in } \Omega. \tag{1.1.1}$$

Here, Ω is a domain in $\mathbb{R}^n$, $n \geq 2$, u is unknown function, $u_{,i} = \partial u/\partial x_i$, $a = (a_{ij})$ is a given symmetric matrix field, $b = (b_i)$ and $g = (g_i)$ are given vector valued functions, c and f are given scalar functions, and summation over repeated indices running from 1 to n is adopted. It is assumed that the matrix a satisfies the ellipticity (uniform ellipticity) condition

$$\nu I \leq a \leq \nu^{-1} I \quad (\Leftrightarrow \nu|\xi|^2 \leq \xi \cdot a\xi \leq \nu^{-1}|\xi|^2 \quad \forall \xi \in \mathbb{R}^n)$$

with a positive constant ν. Here, I is the identity matrix.

Equations (1.1.1) can be re-written in the following invariant form

$$-\mathrm{div}(a\nabla u) + b \cdot \nabla u + cu = f + \mathrm{div}\, g \qquad \text{in } \Omega.$$

In general, equation (1.1.1) may have infinitely many solutions. To select a particular one, a boundary condition should be imposed on. An important example is the Dirichlet boundary condition:

$$u = u_0 \qquad \text{on } \partial\Omega. \tag{1.1.2}$$

Problem (1.1.1), (1.1.2) is called the Dirichlet boundary value problem. A classical solution to (1.1.1), (1.1.2) is a solution u, belonging to $C^2(\Omega)\cap C(\overline{\Omega})$.

The existence and the multiplicity of solutions to boundary value problem (1.1.1), (1.1.2) are important issues in the theory of PDE's. Unfortunately, it might happen that there is no classical solution at all, in particular, if a, b, c, f, and g are not smooth enough.

Several warning messages, indicating that the classical approach to the aforesaid problems does not work, came from physics and the Calculus of Variations in the first part of 20th century. In particular, to prove the existence of a minimizer, say, of the multiple integral

$$\int\limits_{\Omega} F(\nabla u)dx,$$

one has to extend it to non-smooth functions and thus to assume that solutions to the Euler-Lagrange equation for the integrand F are not necessary smooth. On the other hand, δ-function (or Dirac function), introduced by physicist P. Dirac, suggests that the notion of functions as well as solutions should be revised.

That time, the main trend was to include PDE problems into the framework of functional analysis. The principal objects in powerful functional analytic schemes are function spaces and operators, acting there. Suitable spaces such as Lebesgue and Sobolev spaces, in which differential operators have reasonable properties, were discovered in the first part of 20th century. Our course can be regarded as an introduction to the theory of spaces of functions, having so-called weak derivatives, and includes the celebrated Sobolev embedding theorems.

Having in hands "good" function spaces, we shall define the notion of weak solutions, re-discovering old ideas of mechanics and the Calculus of Variations. Namely, we replace our differential equation (1.1.1) with the integral identity

$$\mathcal{L}(u,v)=\int\limits_{\Omega}(a_{ij}u_{,j}v_{,i}+b_iu_{,i}v+cuv)dx=\int\limits_{\Omega}(fv-g_iv_{,i})dx,$$

being valid for any test function v that is sufficiently smooth and vanishes in a neighbourhood of the boundary of Ω. If all functions in the above identity are sufficiently smooth, we may integrate by parts there and derive

$$\mathcal{L}(u,v) = \int\limits_{\Omega} (Lu)v dx = \int\limits_{\Omega} (f + \operatorname{div} g)v dx$$

for the same test functions v. The latter identity shows that all classical solutions are weak solutions as well.

It turns out that the existence of weak solutions is a relatively simple consequence of well-known theorems of functional analysis.

Methods of functional analysis give modern and powerful tools to treat problems related to PDE's and, nowadays, it is difficult to imagine modern mathematics of PDE's without them.

Chapter 2

Lebesgue and Sobolev Spaces

2.1 Lebesgue's Spaces

2.1.1 Spaces $\mathcal{L}^p(E)$ and $L^p(E)$

Let $1 \leq p \leq \infty$ and $E \in \mathbb{R}^n$ be measurable. For simplicity, we always assume that Lebesgue's measure of E is finite, i.e., $|E| := \mu(E) < \infty$, although many statements hold true without this restriction.

Define for a measurable function $f : E \to \overline{\mathbb{R}}$

$$\|f\|_{p,E} := \Big(\int\limits_E |f(x)|^p dx \Big)^{\frac{1}{p}}$$

if $p < \infty$ and

$$\|f\|_{\infty,E} := \operatorname*{ess\,sup}_{x \in E} |f(x)|$$

if $p = \infty$. Here,

$$\operatorname*{ess\,sup}_{x \in E} |f(x)| := \inf\{c > 0 : \ |f| \leq c \text{ a.e. in } E\}.$$

We let then

$$\mathcal{L}^p(E) := \{f : \ f \text{ is measurable and } \|f\|_{p,E} < \infty\}.$$

$\mathcal{L}^p(E)$ is a vector space. Indeed, if f, g are measurable in E and $\lambda \in \mathbb{R}$, then $f + \lambda g$ is measurable. In addition, $|f + \lambda g|^p \leq 2^{p-1}(|f|^p + |\lambda|^p |g|^p)$ for $1 \leq p < \infty$. So, $f + \lambda g \in \mathcal{L}^p(E)$.

Lemma 1.1. *(Hölder inequality) Let $f \in \mathcal{L}^p(E)$, and $g \in \mathcal{L}^{p'}(E)$ with $1/p + 1/p' = 1$. Then $fg \in \mathcal{L}^1(E)$ and*

$$\Big|\int\limits_E f(x)g(x)dx\Big| \le \Big(\int\limits_E |f(x)|^p dx\Big)^{\frac{1}{p}}\Big(\int\limits_E |g(x)|^{p'} dx\Big)^{\frac{1}{p'}}.$$

PROOF Sheet 1.

With the help of Hölder inequality, we can easily prove the following version of Theorem 4.1 of Appendix B on dominated convergence.

Theorem 1.2. *Let f_m, $m = 1, 2, \ldots$, be a sequence of measurable functions in E. Suppose that*
(i) $f_m \to f$ a.e. in E;
(ii) $\sup_m \|f_m\|_{p,E} < \infty$ for some $p > 1$.
Then $f \in \mathcal{L}^p(E)$ and $\|f - f_m\|_{q,E} \to 0$ as $m \to \infty$ and for any $1 \le q < p$.

PROOF By Fatou's lemma, $f \in \mathcal{L}^p(E)$. Let $\gamma > 0$ and $E_m = \{x \in E : |f_m(x) - f(x)| \ge \gamma\}$. By Theorem 2.3 (Lebesgue) of Appendix B, $|E_m| \to 0$ as $m \to \infty$. By Hölder inequality,

$$\|f - f_m\|^q_{q,E} = \|f - f_m\|^q_{q,E\setminus E_m} + \|f - f_m\|^q_{q,E_m} \le$$

$$\le \gamma^q |E \setminus E_m| + |E_m|^{(1-\frac{q}{p})}\|f - f_m\|^q_{p,E_m} \le$$

$$\le \gamma^q|E| + |E_m|^{(1-\frac{q}{p})} c(p,q) \sup_m \|f_m\|^q_{p,E}.$$

Passing to the limit as $m \to \infty$, we find

$$\limsup_{m\to\infty} \|f - f_m\|_{q,E} \le \gamma |E|^{\frac{1}{q}}.$$

Letting γ tend to 0, we complete the proof. □

Lemma 1.3. *(Minkowski inequality)*

$$\Big(\int\limits_E |f(x) + g(x)|^p dx\Big)^{\frac{1}{p}} \le \Big(\int\limits_E |f(x)|^p dx\Big)^{\frac{1}{p}} + \Big(\int\limits_E |g(x)|^p dx\Big)^{\frac{1}{p}}.$$

Proof Sheet 1.

Note $\|f\|_{p,E} = 0$ implies $f = 0$ a.e. in E and thus, from Lemma 1.3, it follows that $\|f\|_{p,E}$ is a semi-norm in $\mathcal{L}^p(E)$.

To work with semi-norms is inconvenient and, in order to avoid this, we introduce equivalence classes in the following natural way. Two measurable functions f and g are equivalent in E ($f \sim g$) if $f = g$ a.e. in E. An equivalence class generated by a measurable function f is denoted by $[f]$. The space of all equivalence classes whose representatives are integrable with power p is denoted by $L^p(E)$. However, in what follows, we shall denote an equivalence class $[f]$ by the function that generates it, i.e., simply by f. Null element of $L^p(E)$ consists of all functions that are equal to zero a.e. in E and, hence, $L^p(E)$ is a normed space.

Theorem 1.4. *$L^p(E)$ is a Banach space.*

Proof (for $1 \leq p < \infty$, $p = \infty$ is an exercise). Let f_m, $m = 1, 2, ...$, be a Cauchy sequence in $L^p(E)$. In particular, this implies

$$\varepsilon_m = \sup_{k>m} \|f_k - f_m\|_{p,E} \to 0$$

as $m \to \infty$. One can find a subsequence ε_{m_s} such that $\varepsilon_{m_s} < 2^{-s}$, $s = 1, 2, ...$ Then, by Hölder inequality,

$$\|f_{m_{s+1}} - f_{m_s}\|_{1,E} \leq |E|^{\frac{1}{p'}} \|f_{m_{s+1}} - f_{m_s}\|_{p,E} \leq \varepsilon_{m_s} |E|^{\frac{1}{p'}}$$

and thus the series

$$\int\limits_E |f_{m_1}| dx + \sum_{s=1}^{\infty} \int\limits_E |f_{m_{s+1}} - f_{m_s}| dx$$

converges. By Beppo Levi theorem (see Theorem 4.2) of Appendix B, the series

$$|f_{m_1}| + \sum_{s=1}^{\infty} |f_{m_{s+1}} - f_{m_s}|$$

converges a.e. in E and the series

$$f_{m_1} + \sum_{s=1}^{\infty} (f_{m_{s+1}} - f_{m_s})$$

convergence a.e. in E absolutely. The later means that

$$S_j = f_{m_1} + f_{m_2} - f_{m_1} + ... + f_{m_{j+1}} - f_{m_j} = f_{m_{j+1}}$$

converges a.e. in E to a measurable function f. So, for sufficiently large j, we have

$$\varepsilon_m^p \geq \int_E |f_{m_j} - f_m|^p dx$$

and $|f_{m_j} - f_m|^p \to |f - f_m|^p$ a.e. in E. By Fatou's lemma (see Lemma 4.3) of Appendix B,

$$\varepsilon_m^p \geq \int_E |f - f_m|^p dx$$

for any m. This completes the proof. □

2.1.2 Sets that are dense in $L^p(E)$

Let $T = \{E_j\}_{j=1}^m$ be a *partition* of E, i.e., $E = \bigcup_{j=1}^m E_j$, $E_j \cap E_k = \emptyset$ if $j \neq k$. $f : E \to \mathbb{R}$ is a *simple* function if there exists a partition T such that $f(x) = c_i$ for $x \in E_i$.

Theorem 1.5. *Let $1 \leq p \leq \infty$. The set of all simple functions in E is dense in $L^p(E)$.*

Proof The proof is based on Lebesgue's partition, on Sheet 1.

Theorem 1.6. *Let $1 \leq p < \infty$. $C(\mathbb{R}^n)$ is dense in $L^p(E)$.*

We start with two auxiliary lemmata.

Lemma 1.7. *Let $A \subset \mathbb{R}^n$ and $\varrho(x, A) = \inf_{z \in A} |x - z|$. Then $|\varrho(x, A) - \varrho(y, A)| \leq |x - y|$.*

Proof We have for $z \in A$

$$\varrho(x, A) \leq |x - z| \leq |x - y| + |y - z|,$$

which implies $\varrho(x, A) \leq |x - y| + \varrho(y, A)$. Replacing x with y and y with x, we complete the proof. □

Lemma 1.8. *Let $1 \le p < \infty$ and $E_0 \subseteq E$ be two measurable sets in $\mathbb{R}^n$. Given $\varepsilon > 0$, there exists a function $g \in C(\mathbb{R}^n)$ such that $\|\chi_{E_0} - g\|_{p,E} < \varepsilon$, where $\chi_{E_0}(x) = 1$ if $x \in E_0$ and $\chi_{E_0}(x) = 0$ if $x \in E \setminus E_0$.*

Proof Since E_0 is measurable, there exist a closed set $F \subseteq E_0$ and an open set $\mathcal{O} \supseteq E_0$ such that $|\mathcal{O} \setminus F| < (\varepsilon/2)^p$. We let

$$0 \le g(x) := \frac{\varrho(x, \mathbb{R}^n \setminus \mathcal{O})}{\varrho(x, F) + \varrho(x, \mathbb{R}^n \setminus \mathcal{O})} \le 1$$

for $x \in \mathbb{R}^n$. Obviously, g is continuous function in $\mathbb{R}^n$. And

$$\|\chi_{E_0} - g\|_{p,E} \le 2|\mathcal{O} \setminus F|^{\frac{1}{p}} \le \varepsilon. \quad \square$$

Proof of Theorem 1.6 (Sheet 1. Hint: First approximate a function by simple functions and then approximate simple functions by continuous functions)

Corollary 1.9. *$L^p(E)$ is separable if $1 \le p < \infty$.*

Indeed, let Q be an open cube such that $E \subseteq Q$. Since $C(\overline{Q})$ is separable, we find a countable set $\{f_k\}_{k=1}^{\infty} \subset C(\overline{Q})$ that is dense in $C(\overline{Q})$ with respect to L^∞-norm. By Theorem 1.6, given $\varepsilon > 0$ and given $f \in L^p(E)$, there exist a function $g \in C(\overline{Q})$ such that $\|f - g\|_{p,E} \le \varepsilon/2$ and a function f_i such that $\|g - f_i\|_{p,C(\overline{Q})} \le \|g - f_i\|_{\infty,C(\overline{Q})} |Q|^{\frac{1}{p}} < \varepsilon/2$. So, $\|f - f_i\|_{p,E} < \varepsilon$. $\square$

However, $L^\infty(E)$ is not separable (on problem sheet). Arguments are as follows. Let T be a countable partition of E, i.e., $T = \{E_j\}_{j=1}^{\infty}$, $E = \bigcup_{j=1}^{\infty} E_j$, $E_i \cap E_j = \emptyset$ if $i \ne j$. We fix it. Define $X_0 \subset L^\infty(E)$ so that $f \in X_0$ if and only if $f \in L^\infty(E)$ and $f(x) = c_i$ for $x \in E_i$, $i = 1, 2, \ldots$. The mapping $\pi : X_0 \to l^\infty$ defined by $\pi f = c = (c_i)$ is an isometric isomorphism. If $L^\infty(E)$ is separable, then X_0 is separable as well (Explain why, see remark below). But this implies separability of l^∞, which is wrong. $\square$

Remark 1.10. *Let $(X, \|\cdot\|)$ be a separable normed space and X_0 be a subset of X. Then X_0 is separable.*

Indeed, there exists a countable set $\{x_k\}_{k=1}^{\infty}$ that is dense in X. Let $\varepsilon_m > 0$ tend to zero as m goes to ∞. We can find $z_{km} \in X_0$ such that

$$\|x_k - z_{km}\| < \varepsilon_m/3 + \varrho(x_k, X_0).$$

To show that $\{z_{km}\}$ is dense in X_0, take an arbitrary ε and sufficiently large m so that $\varepsilon_m < \varepsilon$. Now, let $x \in X_0$. First, we can find x_k such that $\|x - x_k\| < \varepsilon/3$. We have

$$\|x - z_{km}\| \le \|x - x_k\| + \|x_k - z_{km}\| < \varepsilon/3 + \varepsilon_m/3 + \varrho(x_k, X_0) \le$$

$$\le 2\varepsilon/3 + \|x_k - x\| < \varepsilon. \ \square$$

Theorem 1.11. *(integral continuity or continuity of translations) Let E be a measurable bounded set of $\mathbb{R}^n$ and $1 \le p < \infty$. Let $f \in L^p(E)$ be extended by zero from E to the whole $\mathbb{R}^n$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|f(\cdot + h) - f(\cdot)\|_{p,E} := \Big(\int_E |f(x+h) - f(x)|^p dx \Big)^{\frac{1}{p}} < \varepsilon$$

whenever $|h| < \delta$.

Proof We fix a large cube Q such that $E + h \subset Q$ for any $h \in \mathbb{R}^n$ such that $|h| \le 1$. Since $f \in L^p(Q)$, given $\varepsilon > 0$, there exists a function $g \in C(\overline{Q})$ such that $\|f - g\|_{p,Q} < \varepsilon$. Since g is uniformly continuous in $\overline{Q}$, there exists $0 < \delta < 1$ such that $|g(x+h) - g(x)| < \varepsilon |E|^{-\frac{1}{p}}$ as long as $x, x+h \in \overline{Q}$ and $|h| < \delta$. So,

$$\|f(\cdot + h) - f(\cdot)\|_{p,E} \le \|f - g\|_{p,E+h} + \|g(\cdot + h) - g(\cdot)\|_{p,E} + \|g - f\|_{p,E}$$

$$\le 2\|f - g\|_{p,Q} + \|g(\cdot + h) - g(\cdot)\|_{\infty,E} |E|^{\frac{1}{p}} < 3\varepsilon. \ \square$$

2.1.3 Linear Functionals and Weak Convergence in $L^p(E)$

Lemma 1.12. *Let $f \in L^{p'}(E)$. Then $\|f\|_{p',E} = I$, where*

$$I := \sup \Big\{ \int_E f(x) g(x) dx : \ \|g\|_{p,E} = 1 \Big\}.$$

Proof By Hölder inequality, $I(g) := \int_E f(x)g(x)dx \le \|f\|_{p',E}$ if $\|g\|_{p,E} = 1$ and thus $I \le \|f\|_{p',E}$.

Consider first the case $1 < p \le \infty$. Define

$$g_0(x) = \text{sign}f(x)|f(x)|^{p'-1}/\|f\|_{p',E}^{\frac{p'}{p}} \quad (g_0(x) = \text{sign}f(x) \quad \text{if } p = \infty).$$

It is easy check that $\|g_0\|_{p,E} = 1$ and $I \ge I(g_0) = \|f\|_{p',E}$. This completes the proof for the case $1 < p \le \infty$.

If $p = 1$, then, given $\varepsilon > 0$, define a set $E_\varepsilon = \{x \in E : |f(x)| \ge \|f\|_{\infty,E} - \varepsilon\}$ and a function $g_0(x) = \chi_{E_\varepsilon}(x)\text{sign}f(x)/|E_\varepsilon|$. Simple calculations show that $\|g_0\|_{1,E} = 1$ and $I \ge I(g_0) \ge \|f\|_{\infty,E} - \varepsilon$. Passing $\varepsilon \to 0$, we get the statement of the lemma for $p = 1$. □

Theorem 1.13. *(Riesz) Let* $1 \le p < \infty$. *There exists isometric isomorphism* $\pi : (L^p(E))^* \to L^{p'}(E)$, $p' = \frac{p}{p-1}$, *so that* $\pi T = f$ *with*

$$T(g) = \int_E f(x)g(x)dx \qquad \forall g \in L^p(E). \tag{2.1.1}$$

So, we have $(L^p(E))^* \cong L^{p'}(E)$.

From the Riesz representation theorem, it follows that the space $L^p(E)$ is reflexive provided $1 < p < \infty$.

Proposition 1.14. *Assume that*

$$\sup_m \|f_m\|_{p,E} < \infty.$$

If $1 < p < \infty$, *there exists a subsequence* f_{m_k} *such that*

$$f_{m_k} \rightharpoonup f$$

as $k \to \infty$, *i.e., for any* $g \in L^{p'}(E)$,

$$\int_E f_{m_k} g dx \to \int_E f g dx \tag{2.1.2}$$

as $k \to \infty$.
If $p = \infty$ *there exists a subsequence* f_{m_k} *such that*

$$f_{m_k} \overset{*}{\rightharpoonup} f,$$

i.e., (2.1.2) holds true for any $g \in L^1(E)$.

Proof It is a direct consequence of Theorems 6.4 and 6.5 of Appendix A and Theorem 1.13.

2.1.4 Mollification in $L^p(\Omega)$

Define a function for non-negative t as $h(t) = 0$ if $t \geq 1$ and $h(t) = \exp\{\frac{1}{t-1}\}$ if $0 \leq t < 1$. We also let $\omega_1(x) = c_1(n)h(|x|^2)$, $x \in \mathbb{R}^n$, with constant c_1 chosen so that

$$\int_{\mathbb{R}^n} \omega_1(x)dx = 1.$$

For positive ϱ, we define a *mollifier* as $\omega_\varrho(x) = \frac{1}{\varrho^n}\omega_1(\frac{x}{\varrho})$ that is an infinitely differentiable function and equal to 0 outside $B(\varrho)$. By scaling and by shift, we have for any $x \in \mathbb{R}^n$:

$$\int_{\mathbb{R}^n} \omega_\varrho(x-y)dy = \int_{B(x,\varrho)} \omega_\varrho(x-y)dy = 1.$$

Let Ω be a domain in $\mathbb{R}^n$ and $f \in L^1(\Omega)$ be extended by zero to the whole $\mathbb{R}^n$. A mollification of f is

$$f_\varrho(x) := (\omega_\varrho * f)(x) = \int_\Omega \omega_\varrho(x-y)f(y)dy.$$

It is an infinitely differentiable function in $\mathbb{R}^n$ (explain why) and vanishes outside $\Omega^\varrho := \{x \in \mathbb{R}^n : \ \mathrm{dist}(x,\Omega) < \varrho\}$.

Our aim is to show that we can approximate functions from $L^p(\Omega)$ with the help of mollification.

Lemma 1.15. *Let $f \in L^p(\Omega)$ and $1 \leq p \leq \infty$.*

$$\|f_\varrho\|_{p,\Omega} \leq \|f\|_{p,\Omega}.$$

PROOF Let $1 < p < \infty$. We apply Hölder inequality in the following way

$$|f_\varrho(x)| \leq \int_\Omega \omega_\varrho^{\frac{1}{p'}+\frac{1}{p}}(x-y)|f(y)|dy$$

$$= \int_\Omega \omega_\varrho^{\frac{1}{p'}}(x-y)\omega_\varrho^{\frac{1}{p}}(x-y)|f(y)|dy$$

$$\leq \Big(\int\limits_{\Omega} \omega_\varrho(x-y)dy\Big)^{\frac{1}{p'}} \Big(\int\limits_{\Omega} \omega_\varrho(x-y)|f(y)|^p dy\Big)^{\frac{1}{p}}.$$

We know

$$\int\limits_{\Omega} \omega_\varrho(x-y)dy \leq 1.$$

It remains to apply Tonelli's theorem and conclude

$$\|f_\varrho\|^p_{p,\Omega} = \int\limits_{\Omega} |f_\varrho(x)|^p dx \leq \int\limits_{\Omega} dx \int\limits_{\Omega} \omega_\varrho(x-y)|f(y)|^p dy$$

$$= \int\limits_{\Omega} |f(y)|^p dy \int\limits_{\Omega} \omega_\varrho(x-y)dx \leq \int\limits_{\Omega} |f(y)|^p dy = \|f\|^p_{p,\Omega}.$$

Cases $p = 1$ and $p = \infty$ are considered in the same way (explain why). □

Theorem 1.16. *Let $f \in L^p(\Omega)$ and $1 \leq p < \infty$. Then $f_\varrho \to f$ in $L^p(\Omega)$ as $\varrho \to 0$.*

Proof For any $x \in \Omega$, we have (f is extended by zero outside Ω)

$$f_\varrho(x) - f(x) = \int\limits_{\Omega} \omega_\varrho(x-y)f(y)dy - f(x) =$$

$$= \int\limits_{B(x,\varrho)} \omega_\varrho(x-y)f(y)dy - f(x) = \int\limits_{B(x,\varrho)} \omega_\varrho(x-y)(f(y)-f(x))dy$$

and thus

$$|f_\varrho(x) - f(x)| \leq \int\limits_{B(x,\varrho)} \omega_\varrho(x-y)|f(y)-f(x)|dy$$

$$\stackrel{z=y-x}{=} \int\limits_{B(\varrho)} \omega_\varrho(z)|f(x+z)-f(x)|dz.$$

Repeating arguments used in the proof of Lemma 1.15, we find

$$\int\limits_{\Omega} |f_\varrho(x)-f(x)|^p dx \leq \int\limits_{\Omega} dx \int\limits_{B(\varrho)} \omega_\varrho(z)|f(x+z)-f(x)|^p dz$$

$$= \int\limits_{B(\varrho)} \omega_\varrho(z)dz \int\limits_\Omega |f(x+z) - f(x)|^p dx$$

and, therefore,

$$\|f_\varrho - f\|_{p,\Omega} \leq \sup_{|z|<\varrho} \|f(\cdot + z) - f(\cdot)\|_{p,\Omega} \tag{2.1.3}$$

By Theorem 1.11, the right hand side of (2.1.3) tends to 0 as $\varrho \to 0$. $\square$

Theorem 1.17. *(Riesz) Let $\mathcal{F}$ be a subset in $L^p(\Omega)$. Let $1 \leq p < \infty$. $\mathcal{F}$ is precompact in $L^p(\Omega)$ if and only if*
(i) $\sup\limits_{f \in \mathcal{F}} \|f\|_{p,\Omega} =: M < \infty$
(ii) $\sup\limits_{f \in \mathcal{F}} \sup\limits_{|z|<\varrho} \|f(\cdot + z) - f(\cdot)\|_{p,\Omega} =: \delta(\varrho) \to 0$ *as* $\varrho \to 0$.

Proof We start with a proof of *sufficient* conditions of compactness. Let us denote by $\mathcal{F}_\varrho$ the set of all f_ϱ with $f \in \mathcal{F}$ and show that for each fixed $\varrho > 0$ this family satisfies all conditions of Ascoli-Arzela theorem, see Theorem 4.7 of Appendix A. First, this family is uniformly bounded, i.e.,

$$\sup_{x \in \overline{\Omega}} |f_\varrho(x)| \leq c(\varrho, |\Omega|)\|f\|_{p,\Omega} \leq cM.$$

It is equi-continuous, since

$$|f_\varrho(x_1) - f_\varrho(x_2)| \leq \int\limits_\Omega |\omega_\varrho(x_1 - y) - \omega_\varrho(x_2 - y)| f(y)|dy$$

$$\leq c(\varrho, |\Omega|)|x_1 - x_2|\|f\|_{p,\Omega} \leq cM|x_1 - x_2|$$

for any x_1 and x_2 from $\overline{\Omega}$.

We take an arbitrary $\varepsilon > 0$ and choose $\varrho > 0$ so that $\delta(\varrho) < \varepsilon/2$ and fix it. For this $\varrho > 0$, the family $\mathcal{F}_\varrho$ is precompact in $C(\overline{\Omega})$. By Hausdorff's theorem, see Theorem 4.5 of Appendix A, there exists a finite $\varepsilon/(2|\Omega|^{\frac{1}{p}})$-net $\{h_j \in C(\overline{\Omega})\}_{j=1}^m$ and we are going to show that it is ε-net for $\mathcal{F}$. Indeed, for $(f)_\varrho \in \mathcal{F}_\varrho$, there exists h_j such that

$$\|f_\varrho - h_j\|_{p,\Omega} \leq |\Omega|^{\frac{1}{p}} \|f_\varrho - h_j\|_{\infty,\Omega} < \varepsilon/2$$

and thus, by (2.1.3) and by our choice of $\varrho > 0$,

$$\|f - h_j\|_{p,\Omega} \leq \|f - f_\varrho\|_{p,\Omega} + \|f_\varrho - h_j\|_{p,\Omega} < \varepsilon.$$

Now, we are in a position to prove *necessary* conditions. Let Q be a large cube so that $\Omega + h \subset Q$ for any $|h| \leq 1$. Let $\mathcal{F}_Q$ consists of all functions $f \in L^p(Q)$ such that $f(x) = g(x)$ if $x \in \Omega$ for some $g \in \mathcal{F}$ and $f(x) = 0$ if $x \in Q \setminus \Omega$. Obviously, if $\mathcal{F}$ is precompact in $L^p(\Omega)$, then $\mathcal{F}_Q$ is precompact in $L^p(Q)$.

By Hausdorff's theorem, there exists a finite 1-net of $\mathcal{F}_Q$, say, f_j, $j = 1, 2, ..., m$. Then by the definition, given $f \in \mathcal{F}_Q$, there exists f_j from this 1-net such that $\|f - f_j\|_{p,Q} \leq 1$ and thus we have

$$\|f\|_{p,\Omega} < \|f_j\|_{p,Q} + 1 \leq \sup_{1\leq j\leq m} \|f_j\|_{p,Q} + 1 =: M$$

for any $f \in \mathcal{F}$. So, uniform boundedness is proved.

Next, for an arbitrary $\varepsilon > 0$, we have ε-net of $\mathcal{F}_Q$, say, f_j, $j = 1, 2, ..., m$. So, for $f \in \mathcal{F}_Q$, there exists f_j from ε-net of $\mathcal{F}_Q$ such that $\|f - f_j\|_{p,Q} < \varepsilon$. Then, for $|h| < \varrho$.

$$\begin{aligned}
&\|f(\cdot + h) - f(\cdot)\|_{p,\Omega} \leq \\
&\leq \|f(\cdot + h) - f_j(\cdot + h)\|_{p,\Omega} + \|f_j(\cdot + h) - f_j(\cdot)\|_{p,\Omega} + \|f_j - f\|_{p,\Omega} = \\
&= \|f - f_j\|_{p,\Omega+h} + \|f_j(\cdot + h) - f_j(\cdot)\|_{p,\Omega} + \|f_j - f\|_{p,\Omega} \leq \\
&\leq 2\|f_j - f\|_{p,Q} + \sup_{1\leq j\leq m} \sup_{|h|<\varrho} \|f_j(\cdot + h) - f_j(\cdot)\|_{p,\Omega} \leq \\
&\leq 2\varepsilon + \sup_{1\leq j\leq m} \sup_{|h|<\varrho} \|f_j(\cdot + h) - f_j(\cdot)\|_{p,\Omega}.
\end{aligned}$$

It follows from Theorem 1.11, that the second term on the right hand side of the latter inequality tends to 0 as $\varrho \to 0$. So,

$$\limsup_{\varrho\to 0} \sup_{f\in\mathcal{F}} \sup_{|h|<\varrho} \|f(\cdot + h) - f(\cdot)\|_{p,\Omega} \leq 2\varepsilon.$$

By arbitrariness of ε,

$$\lim_{\varrho\to 0} \sup_{f\in\mathcal{F}} \sup_{|h|<\varrho} \|f(\cdot + h) - f(\cdot)\|_{p,\Omega} = 0.\square$$

2.2 Distributions

2.2.1 Spaces of Differentiable Functions

Definition 2.1. *A n-dimensional vector $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) = (\alpha_i)$ with non-negative integer components is a multi-index of order n. $|\alpha| = \alpha_1+\alpha_2+...+\alpha_n$ is a length of α.*

For $x \in \mathbb{R}^n$, we let $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n} = \prod_{i=1}^n x_i^{\alpha_i}$. Summation over multi-indices is denoted as

$$\sum_{|\alpha|\le k} a_\alpha := \sum_{i=0}^{k} \sum_{|\alpha|=i} a_\alpha = \sum_{i=0}^{k} \sum_{\alpha_1+\alpha_2+...+\alpha_n=i} a_{\alpha_1,\alpha_2,...,\alpha_n}.$$

In this notation, a polynomial of order k of n variables can be written as $P_k(x) := \sum_{|\alpha|\le k} p_\alpha x^\alpha$.

Denote $D_i = \partial/\partial x_i$, $D_i f = \partial f/\partial x_i$, $D_i^k = \partial^k/\partial x_i^k$ and introduce a formal n-dimensional vector $D = (D_1, D_2, ..., D_n)$ so that

$$D^\alpha f = D_1^{\alpha_1} D_2^{\alpha_2} ... D_n^{\alpha_n} f = \frac{\partial^{\alpha_1+\alpha_2+...\alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}} f.$$

Exercise Prove that $D^\alpha D^\beta f = D^\beta D^\alpha f$.

Let Ω be a domain (open connected set) in $\mathbb{R}^n$. For bounded Ω, $C^k(\overline{\Omega})$ is a B-space with norm

$$\|f\|_{C^k} := \sum_{|\alpha|\le k} \max_{x\in\overline{\Omega}} |D^\alpha f(x)|.$$

For $f : \mathbb{R}^n \to R$, define supp$f := \overline{\{x \in \mathbb{R}^n : \ |f(x)| > 0\}}$. We say that f is compactly supported in Ω if supp$f$ is compact and contained in Ω. By definition, $C_0^k(\Omega)$ consists of all $f : \mathbb{R}^n \to R$ being continuously differentiable up to order k and having a compact support in Ω. The important case is a linear space $C_0^\infty(\Omega)$ consisting of all infinitely differentiable functions compactly supported in Ω. For example, the function $h(t) = 0$ if $|t| \ge 1$ and $h(t) = \exp\{\frac{1}{t^2-1}\}$ if $|t| < 1$ is of $C_0^\infty(\mathbb{R})$.

2.2.2 Distributions

Definition 2.2. $\mathcal{D}(\Omega)$ *is the space of test functions consisting of all functions from* $C_0^\infty(\Omega)$. *It is endowed with the following notion of convergence. Let* $\varphi_m \in C_0^\infty(\Omega)$, $m = 1, 2, ...$, *and* $\varphi \in C_0^\infty(\Omega)$, *we say that* $\varphi_m \to \varphi$ *in* $\mathcal{D}(\Omega)$ *as* $m \to \infty$ *if there exists a compact* $K \subset \Omega$ *such that supp*$\varphi_m \subseteq K$ $(\forall m = 1, 2, ...)$, *supp*$\varphi \subseteq K$, *and* $D^\alpha \varphi_m \to D^\alpha \varphi$ *uniformly in* K *for any multi-indices* α.

Definition 2.3. *A distribution T on Ω is a linear contnuous functional $T : \mathcal{D}(\Omega) \to \mathbb{R}$. The latter means: $\varphi_m \to \varphi$ in $\mathcal{D}(\Omega) \Rightarrow T(\varphi_m) \to T(\varphi)$ in $\mathbb{R}$. The set of all distributions is denoted $\mathcal{D}'(\Omega)$.*

Very often, the action T on $\varphi \in \mathcal{D}(\Omega)$ is also denoted by $< T, \varphi >:= T(\varphi)$.
EXAMPLES:
1. T is a *regular* distribution if there exists a function $f \in L^1_{loc}(\Omega)$ such that $T(\varphi) = \int\limits_{\Omega} f\varphi dx$. A regular distribution is denoted also as $T = T_f$.

Lemma 2.4. *$T_f = T_g$ if and only if $f = g$ a.e. in Ω.*

PROOF $T_f = T_g \Leftrightarrow \int\limits_{\Omega} (f-g)\varphi dx = 0$ for any $\varphi \in C_0^\infty(\Omega)$. The result follows from Lemma 2.5 below. □

Lemma 2.5. *Let $f \in L^1_{loc}(\Omega)$ and $\int\limits_{\Omega} f\varphi dx = 0$ for $\varphi \in C_0^\infty(\Omega)$. Then $f = 0$ in Ω.*

PROOF Without loss of generality, we may assume that Ω is bounded and $f \in L^1(\Omega)$. Define $\Omega_\varrho = \{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) > \varrho\}$. Fix $\varrho_0 > 0$. Then, for $0 < \varrho < \varrho_0$, the function $y \mapsto \omega_\varrho(x-y) \in C_0^\infty(\Omega)$ for $x \in \Omega_{\varrho_0}$. By the assumption, $f_\varrho(x) = 0$ for all $x \in \Omega_{\varrho_0}$. From Theorem 1.16, it follows that $f = 0$ a.e. in Ω_{ϱ_0} and thus in Ω. □
2. A bounded Radon measure T is a distribution satisfying

$$|T(\varphi)| \leq M\|\varphi\|_{\infty,\Omega}$$

for any $\varphi \in \mathcal{D}(\Omega)$. Given $a \in \Omega$, the Dirac δ-function is a distribution $T(\varphi) = \varphi(a)$, $\varphi \in \mathcal{D}(\Omega)$. Let us show that the Dirac δ-function is not a regular distribution. If yes, there exists $f \in L^1_{loc}(\Omega)$ such that $\int_\Omega f\varphi dx = \varphi(a)$ for any $\varphi \in C_0^\infty(\Omega)$. By Lemma 2.5, $f = 0$ a.e. in $\Omega \setminus \{a\}$ and thus $f = 0$ a.e. in Ω, which implies $\varphi(a) = 0$ for any $\varphi \in C_0^\infty(\Omega)$. This is a contradiction. Nevertheless, physicists often use the formal notation $\delta_a(x)$ for "density" so that $\varphi(a) = \int_\Omega \delta_a(x)\varphi(x)dx$. However, this is just formal notation since the right hand side of the last "identity" makes no sense.

Remark 2.6. *We say that a sequence $\{T_j\}$ of $\mathcal{D}'(\Omega)$ converges to T in the sense of distributions if $T_j(\varphi) \to T(\varphi)$ for any $\varphi \in C_0^\infty(\Omega)$. Obviously, T is a linear functional in $\mathcal{D}(\Omega)$ and, moreover, $T \in \mathcal{D}'(\Omega)$, which is a bit more difficult to prove.*

2.2.3 Distributional Derivatives

Any distribution has partial derivatives of any order in the following way. Let T be a distribution on Ω and consider the linear functional $S(\varphi) = (-1)^{|\alpha|}T(D^\alpha\varphi)$, $\varphi \in \mathcal{D}(\Omega)$. It is easy to check that S is continuous on $\mathcal{D}(\Omega)$.

Definition 2.7. *A distributional derivative of a distribution T is a distribution S denoted by $D^\alpha T$, i.e., $S = D^\alpha T$.*

Definition 2.7 is in accordance with the classical notion of partial derivatives. Indeed, suppose that $f \in C^{|\alpha|}(\overline{\Omega})$, then integration by parts gives

$$(-1)^{|\alpha|}T_f(D^\alpha\varphi) = (-1)^{|\alpha|}\int\limits_\Omega fD^\alpha\varphi dx = \int\limits_\Omega D^\alpha f\varphi dx$$

for any $\varphi \in C_0^\infty(\Omega)$. So, we conclude that in this case

$$D^\alpha T_f = T_{D^\alpha f}.$$

EXAMPLE: *Fundamental solution to Laplace's equation.*

Let, for $x \neq 0$,

$$f(x) = \frac{1}{|x|^{n-2}}$$

for $n \geq 3$. Direct calculations shows that

$$\Delta f(x) = 0 \tag{2.2.1}$$

if $x \in \mathbb{R}^n \setminus \{0\}$. Let us find ΔT_f on $\mathbb{R}^n$. By the definition, $\Delta T_f(\varphi) = T_f(\Delta\varphi)$ for all $\varphi \in \mathcal{D}(\Omega)$. Then

$$\Delta T_f(\varphi) = \int\limits_{\mathbb{R}^n} f\Delta\varphi dx = \lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}^n\setminus B(\varepsilon)} f\Delta\varphi dx.$$

So, after double integration by parts, we show

$$\Delta T_f(\varphi) = -\lim_{\varepsilon\to 0}\int\limits_{\partial B(\varepsilon)}(\frac{\partial\varphi}{\partial\nu}f - \frac{\partial f}{\partial\nu}\varphi)dS + \lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}^n\setminus B(\varepsilon)}\varphi\Delta f dx,$$

where ν is the outward unit normal to the surface $\partial B(\varepsilon)$. The last term on the right hand side vanishes by (2.2.1) and thus

$$\Delta T_f(\varphi) = -\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{n-2}}\int\limits_{\partial B(\varepsilon)}\frac{\partial\varphi}{\partial x_i}\nu_i dS - (n-2)\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{n-1}}\int\limits_{\partial B(\varepsilon)}\varphi dS.$$

Since the surface area of the ball $B(\varepsilon)$ is equal to $S_{n-1}\varepsilon^{n-1}$, where S_{n-1} is the surface area of the unit ball in $\mathbb{R}^n$ and since φ is a smooth function, we find after taking the limit as $\varepsilon \to 0$ the following identity

$$\Delta T_f(\varphi) = -(n-2)S_{n-1}\varphi(0). \tag{2.2.2}$$

However, very often, physicists use the classical notation for (2.2.2)

$$-\Delta f(x) = (n-2)S_{n-1}\delta_0(x), \qquad x \in \mathbb{R}^n,$$

just mentioning that the latter relation is understood in the sense of distributions.

2.3 Sobolev Spaces

2.3.1 Weak Derivatives

Let Ω be a domain in $\mathbb{R}^n$.

Definition 3.1. *Let $u \in L^1_{loc}(\Omega)$. A regular distribution T_v is a weak (or Sobolev) derivative of u in Ω if $T_v = D^\alpha T_u$ and classical notation is used $v := D^\alpha u$. In other words, $v \in L^1_{loc}(\Omega)$ is a weak derivative of u in Ω if*

$$\int\limits_\Omega v\varphi dx = (-1)^{|\alpha|}\int\limits_\Omega uD^\alpha\varphi dx, \qquad \forall\varphi \in C_0^\infty(\Omega).$$

If $u \in C^{|\alpha|}(\overline{\Omega})$, then the integration by parts gives:

$$\int\limits_\Omega D^\alpha u\varphi dx = (-1)^{|\alpha|}\int\limits_\Omega uD^\alpha\varphi dx, \qquad \forall\varphi \in C_0^\infty(\Omega).$$

So, by the uniqueness lemma, see Lemma 2.4, u has a weak derivative $D^\alpha u$ that coincides with the corresponding usual (classical) derivative.

Let us list some elementary properties of weak derivatives:
1°. By Lemma 2.4, u has the only weak derivative $D^\alpha u$ (if it does exist).
2°. Let $u \in L^1_{loc}(\Omega)$ have a weak derivative $D^\alpha u$ in Ω and $\Omega' \subset \Omega$. Then u has a weak derivative $D^\alpha u$ in Ω' and this derivative coincides with the restriction of the original weak derivative $D^\alpha u$ to Ω'.
3°. Let $u_1, u_2 \in L^1_{loc}(\Omega)$ and $D^\alpha u_1$, $D^\alpha u_2$ be the corresponding weak derivatives of u_1, u_2 in Ω, respectively. Then, for any $c_1, c_2 \in \mathbb{R}$, $c_1 u_1 + c_2 u_2$ has a weak derivative in Ω and it is equal to $c_1 D^\alpha u_1 + c_2 D^\alpha u_2$.
Proof of 1° − 3° Exercise.

Let $u_m \in L^1_{loc}(\Omega)$, $m = 1, 2, ...$, and $u \in L^1_{loc}(\Omega)$. We say that $u_m \to u$ in $L^1_{loc}(\Omega)$ as $m \to \infty$ if $u_m \to u$ in $L^1(K)$ for each compact $K \subset \Omega$.

Lemma 3.2. *Let $u_m \to u$ in $L^1_{loc}(\Omega)$ and $D^\alpha u_m \to v$ in $L^1_{loc}(\Omega)$. Then $v = D^\alpha u$ in Ω.*

Proof By definition, for each m, we have

$$\int_\Omega D^\alpha u_m \varphi dx = (-1)^{|\alpha|} \int_\Omega u_m D^\alpha \varphi dx, \qquad \forall \varphi \in C_0^\infty(\Omega).$$

Since φ is compactly supported in Ω, we can take a limit for each fixed φ and show

$$\int_\Omega v \varphi dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi dx. \square$$

2.3.2 Mollification of Functions with Weak Derivatives

Lemma 3.3. *Let $u \in L^1(\Omega)$ have a weak derivative $D^\alpha u \in L^1(\Omega)$. Let $x \in \Omega$ and $0 < \varrho < dist(x, \partial\Omega)$. Then*

$$D^\alpha u_\varrho(x) = (D^\alpha u)_\varrho(x).$$

Proof For simplicity only, consider the case $|\alpha| = 1$. By assumptions, $\overline{B}(x, \varrho) \subset \Omega$ and thus the function $\omega_\varrho(x - \cdot) \in C_0^\infty(\Omega)$. Next,

$$u_\varrho(x) = \int_\Omega \omega_\varrho(x - y) u(y) dy$$

and

$$\frac{\partial u_\varrho}{\partial x_k}(x) = \int_\Omega \frac{\partial \omega_\varrho}{\partial x_k}(x - y) u(y) dy.$$

We notice that
$$\frac{\partial \omega_\varrho}{\partial x_k}(x-y) = -\frac{\partial \omega_\varrho}{\partial y_k}(x-y)$$
and, using the definition of weak derivatives, find
$$\frac{\partial u_\varrho}{\partial x_k}(x) = -\int\limits_\Omega \frac{\partial \omega_\varrho}{\partial y_k}(x-y)u(y)dy = \int\limits_\Omega \omega_\varrho(x-y)\frac{\partial u}{\partial y_k}(y)dy = \left(\frac{\partial u}{\partial x_k}\right)_\varrho(x).\square$$

Lemma 3.4. *Let $1 \le p < \infty$ and u and $D^\alpha u$ be in $L^p(\Omega)$. Then $D^\alpha u_\varrho \to D^\alpha u$ in $L^p_{loc}(\Omega)$.*

Proof We know that $u_\varrho \to u$ and $(D^\alpha u)_\varrho \to D^\alpha u$ in $L^p(\Omega)$. By previous lemma, for any compact $K \in \Omega$ and sufficiently small ϱ, $D^\alpha u_\varrho(x) = (D^\alpha u)_\varrho(x)$ for any $x \in K$. This completes our proof. $\square$

Proposition 3.5. *Let $u \in L^1_{loc}(\Omega)$ and all the weak derivatives of the first order vanish. Then u is a constant in Ω.*

Proof Suppose first that Ω is a ball of radius r and $u \in L^1(\Omega)$. Let us show that u is a constant there. Let B be a ball of radius $r-\varepsilon$ with the same center as Ω. For $0 < \varrho < \varepsilon$, by Lemma 3.3,
$$\frac{\partial u_\varrho}{\partial x_k}(x) = \left(\frac{\partial u}{\partial x_k}\right)_\varrho(x) = 0, \qquad \forall x \in B, \qquad k = 1, 2, ..., n,$$
and thus $u_\varrho(x) = c_\varrho$ for $x \in B$. We know that $u_\varrho \to u$ in $L^1(B)$, which implies that u is a constant in B. This constant is in fact independent of ε (explain why). Tending $\varepsilon \to 0$, we get that u is constant in Ω.

From this particular case and from the fact that Ω is connected, we can deduce the statement, noticing that if two balls containing in Ω have an intersection that u is a constant in the union of these balls. $\square$.

Theorem 3.6. *Let $\phi : \widetilde{\Omega} \to \Omega$ be diffeomorphism of class C^1 and let a locally integrable function $x \in \Omega \mapsto u(x)$ have all weak derivatives of the first order in Ω. Then the function $y \in \widetilde{\Omega} \mapsto v(y) = u(\phi(y))$ also has all weak derivatives of the first order calculated according to the classical chain rule, i.e.,*
$$\frac{\partial v}{\partial y_k}(y) = \sum_{i=1}^n \frac{\partial u}{\partial x_i}(x)\Big|_{x=\phi(y)} \frac{\partial x_i}{\partial y_k}(y).$$

Proof On sheet 2.

2.3.3 Sobolev Spaces

Definition 3.7. *Sobolev space $W^{l,p}(\Omega)$ is a vector space of functions that are integrable with power p and have all weak derivatives up to order l also being integrable with power p.*

This is a normed space with respect to the norm

$$\|u\|_{W^{l,p}(\Omega)} = \|u\|_{p,l,\Omega} = \Big(\sum_{|\alpha|\le l} \|D^\alpha u\|^p_{p,\Omega}\Big)^{\frac{1}{p}} = \Big(\sum_{|\alpha|\le l} \int_\Omega |D^\alpha u|^p dx\Big)^{\frac{1}{p}}$$

or equivalently

$$\|u\|_{W^{l,p}(\Omega)} = \sum_{|\alpha|\le l} \|D^\alpha u\|_{p,\Omega}.$$

Theorem 3.8. *$W^{l,p}(\Omega)$ is a B-space.*

Proof Let $\|u_k - u_m\|_{p,l,\Omega} \to 0$ as $k, m \to \infty$. Then $\|D^\alpha u_k - D^\alpha u_m\|_{p,\Omega} \to 0$ for any $|\alpha| \le l$. Since $L^p(\Omega)$ is a Banach space, there exist functions $w^\alpha \in L^p(\Omega)$ such that $\|D^\alpha u_m - w^\alpha\|_{p,\Omega} \to 0$. We let $u = w^0$. By Lemma 3.2, $w^\alpha = D^\alpha u$ for all $0 \le |\alpha| \le l$ and thus $\|u - u_m\|_{p,l,\Omega} \to 0$ as $m \to \infty$. □

For the space $W^{l,2}(\Omega)$, we introduce a special notation setting $H^l(\Omega) = W^{l,2}(\Omega)$. It becomes a Hilbert space with a scalar product

$$(u, v)_{H^l(\Omega)} = \sum_{|\alpha|\le l} \int_\Omega D^\alpha u D^\alpha v dx = \sum_{|\alpha|\le l} (D^\alpha u, D^\alpha v)_{L^2(\Omega)}.$$

Theorem 3.9. *Let $\phi : \widetilde{\Omega} \to \Omega$ be diffeomorphism of class C^l so that the mapping ϕ and all its derivatives up to order l are continuous in the closure $\widetilde{\Omega}$. Moreover, its Jacobian does not change the sign there. Then if $x \in \Omega \mapsto u(x) \in W^{l,p}(\Omega)$ then $y \in \Omega \mapsto v(y) = u(\phi(y)) \in W^{l,p}(\widetilde{\Omega})$ and there exist positive constants c_1 and c_2 depending only on ϕ and its derivatives such that*

$$c_1\|u\|_{p,l,\Omega} \le \|v\|_{p,l,\widetilde{\Omega}} \le c_2\|u\|_{p,l,\Omega}.$$

Proof Easy consequence of chain rule.

Definition 3.10. *For finite p, $W^{l,p}_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in $W^{l,p}(\Omega)$. It is a B-space. We also define $H^l_0(\Omega) = W^{l,2}_0(\Omega)$.*

Lemma 3.11. *Let $\Omega \subset \widetilde{\Omega}$ and $u \in W_0^{l,p}(\Omega)$. Define $\widetilde{u}(x) = u(x)$ if $x \in \Omega$ and $\widetilde{u}(x) = 0$ if $x \in \widetilde{\Omega} \setminus \Omega$. Then $\widetilde{u} \in W_0^{l,p}(\widetilde{\Omega})$ and $\|\widetilde{u}\|_{l,p,\widetilde{\Omega}} = \|u\|_{l,p,\Omega}$.*

PROOF By the definition, there exists a sequence $u_m \in C_0^\infty(\Omega)$ such that $\|u_m - u\|_{l,p,\Omega} \to 0$ as $m \to \infty$. Let us denote by $\widetilde{u}_m$ the extension of u_m by zero to $\widetilde{\Omega}$. Obviously, $\widetilde{u}_m \in C_0^\infty(\widetilde{\Omega})$ and of course $\|\widetilde{u}_m - \widetilde{u}_k\|_{l,p,\widetilde{\Omega}} = \|u_m - u_k\|_{l,p,\Omega}$ and $\|\widetilde{u}_m\|_{l,p,\widetilde{\Omega}} = \|u_m\|_{l,p,\Omega}$. From the first identity, it follows that $\widetilde{u}_m$ is a Cauchy sequence in $W^{l,p}(\widetilde{\Omega})$ and thus $\widetilde{u} \in W_0^{l,p}(\widetilde{\Omega})$. The second identity yields the statement of the lemma. □

Lemma 3.12. *Let $u \in W_0^{l,p}(\Omega)$. Then $u_\varrho \to u$ in $W^{l,p}(\Omega)$ as $\varrho \to 0$.*

PROOF See Problem Sheet 2.

Proposition 3.13. *(integration by parts) Let $u \in W^{l,p'}(\Omega)$ and $v \in W_0^{l,p}(\Omega)$ so that $1/p + 1/p' = 1$ with $1 \le p < \infty$. Then, for any $|\alpha| \le l$,*

$$\int_\Omega v D^\alpha u dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha v dx. \tag{2.3.1}$$

PROOF Let $v_m \in C_0^\infty(\Omega)$ be an approximating sequence for v. Then, by definition of weak derivatives, we have

$$\int_\Omega v_m D^\alpha u dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha v_m dx.$$

We find (2.3.1) by tending m to ∞. □

Lemma 3.14. *(Friedrichs inequality) Let $u \in W_0^{l,p}(\Omega)$. Then*

$$\|u\|_{p,\Omega} \le d^l |u|_{p,l,\Omega}, \tag{2.3.2}$$

where $d = diam\,\Omega$, $|u|_{p,l,\Omega} = \left(\sum_{|\alpha|=l} \|D^\alpha u\|_{p,\Omega}^p\right)^{\frac{1}{p}}$.

PROOF Obviously, it is sufficient to prove (2.3.2) for $u \in C_0^\infty(\Omega)$. Without loss of generality, we may assume that $\Omega \subset Q_n = \{x = (x_i) : \ 0 < x_i < d\}$. We extend u by zero to the cube Q_n. We let $x = (y, x_n)$, where $y = (x_1, x_2, ..., x_{n-1})$. Then

$$u(y, x_n) = \int_0^{x_n} \frac{\partial u}{\partial t}(y, t) dt.$$

By Hölder inequality,

$$|u(y,x_n)| \le \Big(\int_0^{x_n} \Big|\frac{\partial u}{\partial t}(y,t)\Big|^p dt\Big)^{\frac{1}{p}} \Big(\int_0^{x_n} 1^{p'} dt\Big)^{\frac{1}{p'}} \le d^{\frac{1}{p'}} \Big(\int_0^{d} \Big|\frac{\partial u}{\partial t}(y,t)\Big|^p dt\Big)^{\frac{1}{p}}.$$

Integrating the latter inequality over Q_n and applying Tonelli's theorem, we find

$$\int_{Q_n} |u(y,x_n)|^p dx = \int_0^d dx_n \int_{Q_{n-1}} |u(y,x_n)|^p dy \le$$

$$\le d^{\frac{p}{p'}} \int_0^d dx_n \int_{Q_{n-1}} dy \int_0^d \Big|\frac{\partial u}{\partial x_n}(y,t)\Big|^p dt = d^{1+\frac{p}{p'}} \int_{Q_n} \Big|\frac{\partial u}{\partial x_n}(y,x_n)\Big|^p dx.$$

Since $1 + \frac{p}{p'} = p$, we have

$$\Big(\int_{Q_n} |u|^p dx\Big)^{\frac{1}{p}} \le d\Big(\int_{Q_n} \Big|\frac{\partial u}{\partial x_n}\Big|^p dx\Big)^{\frac{1}{p}}.$$

Proceeding in the same way, we show that

$$\Big(\int_{Q_n} \Big|\frac{\partial u}{\partial x_n}\Big|^p dx\Big)^{\frac{1}{p}} \le d\Big(\int_{Q_n} \Big|\frac{\partial^2 u}{\partial x_n^2}\Big|^p dx\Big)^{\frac{1}{p}}$$

and so on. As a result,

$$\Big(\int_{Q_n} |u|^p dx\Big)^{\frac{1}{p}} \le d^l\Big(\int_{Q_n} \Big|\frac{\partial^l u}{\partial x_n^l}\Big|^p dx\Big)^{\frac{1}{p}} \le d^l |u|_{p,l,\Omega}. \square$$

Corollary 3.15. *$\|\cdot\|_{p,l,\Omega}$ and $|\cdot|_{p,l,\Omega}$ are equivalent in $W_0^{l,p}(\Omega)$.*

Our next question is about density of smooth functions in Sobolev spaces. We denote by $\widetilde{W}^{l,p}(\Omega)$ the closure of $C^\infty(\overline{\Omega})$ in $W^{l,p}(\Omega)$. Obviously, $\widetilde{W}^{l,p}(\Omega)$ is a subspace in $W^{l,p}(\Omega)$. Very often, these two spaces coincide and this depends on how "good" or "bad" domain Ω is.

Definition 3.16. *Ω is a star-shaped domain if there exists a point $x_0 \in \Omega$ such that, for any $x \in \Omega$, the line segment, joining x and x_0, is in Ω.*

Theorem 3.17. $\widetilde{W}^{l,p}(\Omega) = W^{l,p}(\Omega)$ *for bounded star-shaped domains.*

PROOF The idea of the proof is as follows. We extend a function $u \in W^{l,p}(\Omega)$ outside of Ω so that the extension slightly differs from the original one and then we approximate the extended function by mollification.

We may assume that $x_0 = 0$.

Lemma 3.18. *Let $f \in L^p(\Omega)$ with $1 \le p < \infty$ be extended by zero to the whole $\mathbb{R}^n$. Then*

$$\|f(\cdot/\lambda) - f(\cdot)\|_{p,\Omega} := \Big(\int\limits_{\Omega} |f(x/\lambda) - f(x)|^p dx \Big)^{\frac{1}{p}} \to 0 \qquad \text{as } \lambda \to 1.$$

PROOF On problem sheet 3, similar to the proof of the integral continuity, see Theorem 1.11.

Let us proceed with a proof of Theorem 3.17. We restrict ourselves to the case $l = 1$. For $\lambda > 1$, we let $\Omega_\lambda = \phi_\lambda(\Omega)$ where $\phi_\lambda(x) = \lambda x$. Define $u_\lambda(x) = u(y)$, setting $x = \lambda y \in \Omega_\lambda$ for $y \in \Omega$. By Theorem 3.9, $u_\lambda \in W^{1,p}(\Omega_\lambda)$ and

$$\frac{\partial u_\lambda}{\partial x_k}(x) = \frac{1}{\lambda}\frac{\partial u}{\partial y_k}(y)\Big|_{y=x/\lambda}, \qquad x \in \Omega_\lambda.$$

By the lemma above,

$$\|u_\lambda - u\|_{p,1,\Omega} \to 0 \qquad \text{as } \lambda \to 1.$$

Now, consider mollification of u_λ, i.e.,

$$(u_\lambda)_\varrho(x) = \int\limits_{\Omega_\lambda} \omega_\varrho(x-y) u_\lambda(y) dy.$$

Obviously, $(u_\lambda)_\varrho \in C^\infty(\overline{\Omega})$ and we know, see Lemma 3.4, that

$$\|(u_\lambda)_\varrho - u_\lambda\|_{p,1,\Omega} \to 0 \qquad \text{as } \varrho \to 0.$$

Given $k \in \mathbb{N}$, we first find $\lambda_k > 1$ such that $\|u_{\lambda_k} - u\|_{p,1,\Omega} < 1/k$ and find $\varrho_k > 0$ such that $\|(u_{\lambda_k})_{\varrho_k} - u_{\lambda_k}\|_{p,1,\Omega} < 1/k$. Letting $u_k := (u_{\lambda_k})_{\varrho_k}$, we deduce from the triangle inequality that $\|u_k - u\|_{p,1,\Omega} < 2/k \to 0$ as $k \to \infty$. So, sequence u_k is required. □

Definition 3.19. *A domain Ω is locally star-shaped if for any $x \in \partial\Omega$, there exists a neighborhood $\mathcal{O}_x$ such that the domain $\Omega \cap \mathcal{O}_x$ is star-shaped.*

Theorem 3.20. *Let Ω be a bounded locally star-shaped domain. Then*

$$\widetilde{W}^{l,p}(\Omega) = W^{l,p}(\Omega)$$

provided $1 \le p < \infty$.

2.3.4 Extension of Functions with Weak Derivatives

Let y be a Cartesian system of coordinates in $\mathbb{R}^n$.

$$C_y(R,h) = \{y = (y', y_n) \in \mathbb{R}^n : \ y' = (y_1, y_2, ..., y_{n-1}),\ |y'| < R,\ |y_n| < h\}$$

is a *right circle cylinder.*

Definition 3.21. *Let x be a Cartesian system of coordinates in $\mathbb{R}^n$. A Lipschitz domain (or domain with Lipschitz boundary) Ω is a domain with the boundary $\partial\Omega$ satisfying the following property. For any $x_0 \in \partial\Omega$, there exist positive numbers L, R, a Cartesian system of coordinates y centred at the point x_0, and a function $\phi : \{|y'| \le R\} \to \mathbb{R}$ such that:*
(i) $\partial\Omega \cap \overline{C}_y(R, 2LR) = \{y \in \mathbb{R}^n : \ y_n = \phi(y'),\ |y'| \le R\}$
(ii) $\overline{\Omega} \cap \overline{C}_y(R, 2LR) = \{y \in \mathbb{R}^n : \ |y'| \le R,\ \phi(y') \le y_n \le 2LR\}$
(iii) function ϕ is Lipschitz continuous with the Lipschitz constant L, i.e., $|\phi(y') - \phi(z')| \le L|y' - z'|$ for $y', z' \in \mathbb{R}^{n-1}$, $|y'| \le R$, $|z'| \le R$.

Remark 3.22. *Numbers R, L, and function ϕ may depend on $x_0 \in \partial\Omega$. Relationship between old (global) and new (local) Cartesian coordinates x and y is given by $y = Q(x - x_0)$ with an orthogonal matrix Q.*

Remark 3.23. *Any Lipschitz domain is locally star-shaped.*

Remark 3.24. *A Lipschitz domain is of class C^k if the function ϕ of Definition 3.21 belongs to $C^k\{|y'| \le R\}$.*

EXAMPLES

Theorem 3.25. *Let Ω be a Lipschitz domain and Ω_0 be a domain such that $\Omega \Subset \Omega_0$. For any $u \in W^{1,p}(\Omega)$, there exists a function $v \in W_0^{1,p}(\Omega_0)$ with the following properties:*
(i) $v(x) = u(x)$, $x \in \Omega$
(ii) $\|v\|_{p,1,\Omega_0} \le c\|u\|_{p,1,\Omega}$ with a constant depending only on n, p, Ω, and Ω_0.

Chapter 3

The First Embedding Theorem

3.1 Sobolev embedding of $W^{1,p}(\Omega)$ into $L^q(\Omega)$

Let B_1 and B_2 be two B-spaces. We say that B_2 is *embedded* into B_1 if $B_2 \subset B_1$. The embedding of B_2 into B_1 is *continuous* if there exists a constant c such that $\|u\|_{B_1} \leq c\|u\|_{B_2}$ for any $u \in B_2$. When talking about embedding, we always keep in mind continuous embedding. Trivial examples are:
(i) $B_1 = L^{p_1}(\Omega)$, $B_2 = L^{p_2}(\Omega)$ for $p_1 < p_2$ if Ω is a bounded domain
(ii) $B_1 = L^p(\Omega)$, $B_2 = W^{1,p}(\Omega)$
Embedding is called *compact* if any set bounded in B_2 is precompact in B_1.

Lemma 1.1. *Let $n > 1$. For any $u \in C_0^1(\mathbb{R}^n)$,*

$$\Big(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx\Big)^{\frac{n-1}{n}} \leq \prod_{i=1}^{n} \Big(\int_{\mathbb{R}^n} |D_i u| dx\Big)^{\frac{1}{n}}, \tag{3.1.1}$$

where $D_i = \partial/\partial x_i$.

Proof Proof is by induction on n. Let $n = 2$. Then

$$u(x_1, x_2) = \int_{-\infty}^{x_1} \frac{\partial u}{\partial t}(t, x_2) dt$$

and thus

$$|u(x_1, x_2)| \leq \int_{-\infty}^{x_1} \Big|\frac{\partial u}{\partial t}(t, x_2)\Big| dt \leq \int_{-\infty}^{\infty} \Big|\frac{\partial u}{\partial x_1}(x_1, x_2)\Big| dx_1 \leq \int_{-\infty}^{\infty} |D_1 u| dx_1.$$

The same arguments give

$$|u(x_1, x_2)| \leq \int_{-\infty}^{\infty} |D_2 u| dx_2.$$

From the latter bounds, it follows that

$$\int_{\mathbb{R}^2} |u|^2 dx \leq \int_{\mathbb{R}^2} dx \int_{-\infty}^{\infty} |D_1 u| dx_1 \int_{-\infty}^{\infty} |D_2 u| dx_2.$$

By Tonelli's theorem the right hand side of the last inequality is

$$\int_{\mathbb{R}^2} |D_1 u| dx \int_{\mathbb{R}^2} |D_2 u| dx.$$

Now, assume that our statement is valid for $n-1$ and let us show its validity for n. In the way as in 2D case, one can prove that

$$|u(x)| \leq \int_{-\infty}^{\infty} |D_i u| dx_i, \qquad i = 1, 2, ..., n. \tag{3.1.2}$$

Next, letting $x' = (x_1, x_2, ..., x_{n-1})$, we have, by Tonelli's theorem,

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx = \int_{-\infty}^{\infty} dx_n \int_{\mathbb{R}^{n-1}} |u||u|^{\frac{1}{n-1}} dx'. \tag{3.1.3}$$

By Hölder inequality and by induction

$$\int_{\mathbb{R}^{n-1}} |u||u|^{\frac{1}{n-1}} dx' \leq \Big(\int_{\mathbb{R}^{n-1}} (|u|^{\frac{1}{n-1}})^{n-1} dx' \Big)^{\frac{1}{n-1}} \Big(\int_{\mathbb{R}^{n-1}} |u|^{\frac{n-1}{n-2}} dx' \Big)^{\frac{n-2}{n-1}} \leq$$

$$\leq \Big(\int_{\mathbb{R}^{n-1}} |u| dx' \Big)^{\frac{1}{n-1}} \prod_{i=1}^{n-1} \Big(\int_{\mathbb{R}^{n-1}} |D_i u| dx' \Big)^{\frac{1}{n-1}}.$$

Using (3.1.3) and (3.1.2), we derive from the last estimate

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx \leq \int_{-\infty}^{\infty} dx_n \Big(\int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} |D_n u| dx' dx_n \Big)^{\frac{1}{n-1}} \prod_{i=1}^{n-1} \Big(\int_{\mathbb{R}^{n-1}} |D_i u| dx' \Big)^{\frac{1}{n-1}} =$$

$$= \Big(\int\limits_{\mathbb{R}^n} |D_n u| dx \Big)^{\frac{1}{n-1}} \int\limits_{-\infty}^{\infty} dx_n \prod_{i=1}^{n-1} \Big(\int\limits_{\mathbb{R}^{n-1}} |D_i u| dx' \Big)^{\frac{1}{n-1}}.$$

Applying Hólder inequality one more time to the second multiplier on the right hand side of the latter inequality, we complete the proof of the lemma. □

Theorem 1.2. *(Gagliardo-Nirenberg inequality) Let* Ω *be an arbitrary domain in* $\mathbb{R}^n$ *and let* $1 \le p < n$*. For any* $u \in W_0^{1,p}(\Omega)$*,*

$$\|u\|_{\overline{p},\Omega} \le \frac{p(n-1)}{n-p} |u|_{p,1,\Omega}, \tag{3.1.4}$$

where $\overline{p} = \frac{np}{n-p}$*.*

PROOF We shall prove the theorem for the case $p > 1$. The case $p = 1$ is an exercise.

We let $\kappa = \frac{n-p}{(n-1)p}$. It is easy to check that $1/\kappa > 1$. For an arbitrary $u \in C_0^\infty(\Omega)$ extended by zero to the whole $\mathbb{R}^n$, define $v = |u|^{\frac{1}{\kappa}}$. Since $D_i v = \frac{1}{\kappa} |u|^{\frac{1}{\kappa}-1} \mathrm{sign}(u) D_i u$, $v \in C_0^1(\mathbb{R}^n)$. By Lemma 1.1,

$$\|v\|_{\frac{n}{n-1},\Omega} \le \prod_{i=1}^{n} \Big(\int\limits_{\Omega} |D_i v| dx \Big)^{\frac{1}{n}}. \tag{3.1.5}$$

After direct calculations, we see

$$\|v\|_{\frac{n}{n-1},\Omega} = \|u\|_{\overline{p},\Omega}^{\frac{1}{\kappa}}.$$

For the right hand side of (3.1.5), we apply Hölder inequality

$$\int\limits_{\Omega} |D_i v| dx = \frac{1}{\kappa} \int\limits_{\Omega} |u|^{\frac{1}{\kappa}-1} |D_i u| dx \le$$

$$\le \frac{1}{\kappa} \Big(\int\limits_{\Omega} |D_i u|^p dx \Big)^{\frac{1}{p}} \Big(\int\limits_{\Omega} (|u|^{\frac{1}{\kappa}-1})^{\frac{p}{p-1}} dx \Big)^{\frac{p-1}{p}} \le \frac{1}{\kappa} |u|_{p,1,\Omega} \|u\|_{\overline{p},\Omega}^{\frac{1}{\kappa}-1}.$$

Now, from (3.1.5) and from two latter bounds, we get required inequality (3.1.4) for $u \in C_0^\infty(\Omega)$.

If $u \in W_0^{1,p}(\Omega)$, then, by the definition, there exists a sequence $u_m \in C_0^\infty(\Omega)$ such that $\|u-u_m\|_{p,1,\Omega} \to 0$ as $m \to \infty$. So, it is a Cauchy sequence in $W^{1,p}(\Omega)$ and, by inequality (3.1.4) for smooth compactly supported functions, is a Cauchy sequence in $L^{\overline{p}}(\Omega)$. Then we finish our proof by taking the limit in (3.1.4) with $u = u_m$. □

Corollary 1.3. *(Poincarè inequality) For any $u \in W_0^{1,2}(\Omega)$,*

$$\|u\|_{2,\Omega} \leq c(n)|\Omega|^{\frac{1}{n}}|u|_{2,1,\Omega}.$$

PROOF Let us find p for which $\overline{p} = 2$. It is $p = \frac{2n}{n+2} < 2$. So, by (3.1.4),

$$\|u\|_{2,\Omega} \leq \frac{2(n-1)}{n}|u|_{\frac{2n}{n+2},1,\Omega} = \frac{2(n-1)}{n}\Big(\int\limits_\Omega \sum_{i=1}^n |D_i u|^{\frac{2n}{n+2}} dx\Big)^{\frac{n+2}{2n}}.$$

It remains to apply Hölder inequality for sums and integrals and complete the proof. □

Theorem 1.4. *(Sobolev) Let Ω be a bounded domain with Lipschitz boundary and let $1 \leq p \leq n$. Then:*
(i) if $1 \leq p < n$ then Sobolev space $W^{1,p}(\Omega)$ is embedded (continuously) into Lebesgue space $L^q(\Omega)$ for any $q \in [1, \frac{pn}{n-p}]$;
(ii) if $p = n$ then Sobolev space $W^{1,p}(\Omega)$ is embedded (continuously) into Lebesgue space $L^q(\Omega)$ for any $1 \leq q < \infty$.

PROOF. Let us fix a bounded domain Ω_0 so that $\Omega \Subset \Omega_0$. Then for any $u \in W^{1,p}(\Omega)$, there exists a function $v \in W_0^{1,p}(\Omega_0)$ such that
(i) $v = u$ in Ω
(ii) $\|v\|_{p,1,\Omega_0} \leq c_1(n, p, \Omega_0, \Omega)\|u\|_{p,1,\Omega}$.

Obviously, by (3.1.4), we have

$$\|u\|_{\overline{p},\Omega} \leq \|v\|_{\overline{p},\Omega_0} \leq \frac{(n-1)p}{n-p}|v|_{p,1,\Omega_0} \leq \frac{(n-1)p}{n-p}\|v\|_{p,1,\Omega_0} \leq$$
$$\leq c_1\frac{(n-1)p}{n-p}\|u\|_{p,1,\Omega}.$$

To finish the proof of the first part of the theorem, it is sufficient to note that $\|u\|_{q,\Omega} \leq |\Omega|^{\frac{1}{q}-\frac{1}{\overline{p}}}\|u\|_{\overline{p},\Omega}$. The second part follows from the first one and obvious continuous embedding $W^{1,p}(\Omega)$ into $W^{1,q}(\Omega)$ provided $1 \leq q < p$. □

Our next question is under which assumptions the above continuous embeddings are compact. We start with the following theorem

Theorem 1.5. *(Rellich-Kondrachov) Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then $W^{1,p}(\Omega)$ is compactly embedded into $L^p(\Omega)$.*

PROOF We shall prove the theorem for finite p.

Let $u \in C_0^\infty(\mathbb{R}^n)$. Fix an arbitrary vector $z \in \mathbb{R}^n$ and define the function $\vartheta(t) = u(x+tz)$. Then

$$u(x+z) - u(x) = \vartheta(1) - \vartheta(0) = \int\limits_0^1 \frac{d\vartheta}{dt}(t)dt,$$

where

$$\frac{d\vartheta}{dt}(t) = \sum_{i=1}^n D_i u(y)|_{y=x+tz} z_i.$$

Next, we have

$$I := \int\limits_{\mathbb{R}^n} |u(x+z) - u(x)|^p dx = \int\limits_{\mathbb{R}^n} dx \Big| \int\limits_0^1 \sum_{i=1}^n D_i u(x+tz) z_i dt \Big|^p.$$

Then we apply Hölder inequality for integrals and sums:

$$I \leq \int\limits_{\mathbb{R}^n} dx \int\limits_0^1 \Big| \sum_{i=1}^n D_i u(x+tz) z_i \Big|^p dt \leq n^{\frac{p}{p'}} \int\limits_{\mathbb{R}^n} dx \int\limits_0^1 \sum_{i=1}^n |D_i u(x+tz)|^p |z_i|^p dt$$

$$\leq n^{p-1} |z|^p \int\limits_0^1 \int\limits_{\mathbb{R}^n} \sum_{i=1}^n |D_i u(x+tz)|^p dx dt.$$

After change of variables $y = x + tz$, we find

$$\|u(\cdot + z) - u(\cdot)\|_{p,\mathbb{R}^n} \leq c_2(n,p)|z||u|_{p,1,\mathbb{R}^n} \qquad (3.1.6)$$

for any $u \in C_0^\infty(\mathbb{R}^n)$.

Now, let U be a bounded set in $W^{1,p}(\Omega)$, i.e., $\|u\|_{p,1,\Omega} \leq M < \infty$ for any $u \in U$. We fix a bounded domain Ω_0 such that $\Omega \Subset \Omega_0$. Define $V \subset W_0^{1,p}(\Omega_0)$ as follows: $v \in V$ if and only if there exists $u \in U$ such that $v = u$ in Ω and $\|v\|_{p,1,\Omega_0} \leq c_3(n,p,\Omega,\Omega_0)\|u\|_{p,1,\Omega}$. So, V is also bounded in $W^{1,p}(\Omega_0)$ and therefore in $L^p(\Omega_0)$. Next, by the definition of $W_0^{1,p}(\Omega_0)$, there exists a sequence $v_m \in C_0^\infty(\Omega_0)$ is converging to v in $W^{1,p}(\Omega_0)$ and thus in $L^p(\Omega_0)$ and even in $L^p(\mathbb{R}^n)$. Since (3.1.6) gives:

$$\|v_m(\cdot + z) - v_m(\cdot)\|_{p,\Omega_0} \le c_2(n,p)|z||v_m|_{p,1,\Omega_0},$$

(3.1.6) is valid for any function $v \in W_0^{1,p}(\Omega_0)$ and, therefore,

$$\|v(\cdot + z) - v(\cdot)\|_{p,\Omega_0} \le \|v(\cdot + z) - v(\cdot)\|_{p,\mathbb{R}^n} \le c_2(n,p)|z||v|_{p,1,\Omega_0} \le c_2 c_3 |z| M$$

for any $v \in V$ and for any $z \in \mathbb{R}^n$. Hence, by Theorem 1.17, the set V is precompact in $L^p(\Omega_0)$ and therefore the set U is precompact in $L^p(\Omega)$. □

Theorem 1.6. *(Sobolev-Kondrachov) Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then $W^{1,p}(\Omega)$ is compactly embedded into $L^q(\Omega)$ with any $1 \le q < \overline{p}$, where $\overline{p} = \frac{np}{n-p}$ if $1 \le p < n$ and $\overline{p} = \infty$ if $p = n$.*

Proof Consider the case $1 \le p < n$ only. The case $p = n$ is an exercise. Let U be a bounded set in $W^{1,p}(\Omega)$ and u_m is an arbitrary sequence in U. By Theorem 1.5, there exists a subsequence u_{m_k} such that $\|u_{m_k} - u\|_{p,\Omega} \to 0$ as $k \to \infty$ for some $u \in W^{1,p}(\Omega)$ and thus $u_{m_k} \to u$ in measure in Ω. On the other hand, by Theorem 1.4, this sequence is bounded in $L^{\overline{p}}(\Omega)$. Then, by Theorem 1.2, we show that $u_{m_k} \to u$ in $L^q(\Omega)$ with $q < \overline{p}$. □

3.2 Traces of functions with weak derivatives

3.2.1 Surface Integral

Let Ω be a bounded domain of class C^1. Going back to Definition 3.21, note that cylinders $C_y(R(x), 2L(x)R(x))$ for $x \in \partial\Omega$ are an open cover of the compact $\partial\Omega$. By the Heine-Borel lemma, there exists a finite subcover, i.e.,

$$\partial\Omega \subset \bigcup_{k=1}^{m} C_{y_{(k)}}(R_k, 2L_k R_k),$$

where $R_k = R(x^{(k)})$, $L_k = L(x^{(k)})$, $y_{(k)} = Q_k(x - x^{(k)})$, and Q_k is an orthogonal matrix. Denote that $S_k = \{|y'_{(k)}| < R_k\} \subset \mathbb{R}^{n-1}$. Under our assumptions, the function ϕ_k, the graph of which is a part of $\partial\Omega$, belongs to $C^1(\overline{S}_k)$. For this subcover, there exists a finite partition of unity $\vartheta_k \in C_0^\infty(\mathbb{R}^n)$ such that $0 \le \vartheta_k \le 1$ and $\operatorname{supp}\vartheta \subset C_{y_{(k)}}(R_k, 2L_k R_k)$, $k = 1, 2, ..., m$, and $\sum_{k=1}^{m} \vartheta_k(x) = 1$ for all $x \in \partial\Omega$.

Definition 2.1. *$f : \partial\Omega \to \overline{\mathbb{R}}$ is Lebesgue integrable in $\partial\Omega$ if the value*

$$I := \sum_{k=1}^{m} \int\limits_{S_k} \vartheta_k(x(y'_{(k)}, \phi_k(y'_{(k)})))f(x(y'_{(k)}, \phi_k(y'_{(k)})))\times$$

$$\times\sqrt{1 + \sum_{i=1}^{n-1} \Big|\frac{\partial \phi_k}{\partial y'_{i(k)}}(y'_{(k)})\Big|^2} dy'_{(k)}.$$

is finite

All the integrals in the above definition are taken with respect to Lebesgue measure in $\mathbb{R}^{n-1}$. If f is integrable in $\partial\Omega$, we shall write

$$\int\limits_{\partial\Omega} f dS := I.$$

One can show that the integral is well-defined, i.e., independent of the choice of subcover. We also can introduce a surface Lebesgue measure and measurable (with respect to this measure) sets as follows. Let $\Gamma \subseteq \partial\Omega$ and χ_Γ be its characteristic function. If χ_Γ is integrable, then the set Γ is called measurable and the corresponding integral is called its surface measure. It is a natural generalisation of the surface area.

We say that $f : \Gamma \subseteq \partial\Omega \to \mathbb{R}$ is integrable in Γ if its extension $\widetilde{f} : \partial\Omega \to \mathbb{R}$ by zero to the whole $\partial\Omega$ is integrable in $\partial\Omega$ and we let

$$\int\limits_{\Gamma} f dS = \int\limits_{\partial\Omega} \widetilde{f} dS.$$

For functions integrable in Γ, the same statements as for functions integrable in $\mathbb{R}^n$ are valid as well. We also can introduce Lebesgue space $L^p(\Gamma)$.

Remark 2.2. *All the statements and constructions remains to be true for domains with Lipschitz boundary.*

3.2.2 Traces of functions from Sobolev Spaces

Lemma 2.3. *Let Ω be a domain of class C^1 and let $1 \le p < \infty$. There exists a constant $c(n, p, \Omega)$ such that*

$$\|u\|_{p,\partial\Omega} \le c\|u\|_{p,1,\Omega}, \qquad \forall u \in C^1(\overline{\Omega}). \tag{3.2.1}$$

Proof We shall consider the case $p > 1$. The case $p = 1$ is an exercise. To avoid technical difficulties and demonstrate the essence of the matter, consider the following particular case. Assume that $\partial\Omega$ contains a flat part $\Gamma = \{x = (x', x_n) : \ |x'| < r, \, x_n = 0\}$ and prove instead of (3.2.1) a simpler inequality

$$\|u\|_{p,\Gamma} \le c\|u\|_{p,1,\Omega}, \qquad \forall u \in C^1(\overline{\Omega}). \tag{3.2.2}$$

Let $h > 0$ be so small that $\Omega_h := \{x = (x', x_n) : \ |x'| < r, \, 0 < x_n < h\} \subset \Omega$. Fix a function $\eta \in C_0^\infty(\mathbb{R})$ so that $\eta(t) = 1$ for $|t| \le h/3$ and $\eta(t) = 0$ for $|t| > 2h/3$. Then we have

$$|u(x', 0)|^p = -\int_0^h \frac{\partial}{\partial t}|\eta(t)u(x', t)|^p dt =$$

$$-p\int_0^h |\eta(t)u(x', t)|^{p-1}\mathrm{sign}(\eta(t)u(x', t))\frac{\partial \eta u}{\partial t}(x', t)dt.$$

After applying Hölder inequality, we find

$$|u(x', 0)|^p \le c_1\Big(\int_0^h |u(x', x_n)|^p dx_n\Big)^{\frac{1}{p'}}\Big(\int_0^h \Big|\frac{\partial u}{\partial x_n}(x', x_n)\Big|^p dx_n\Big)^{\frac{1}{p}} +$$

$$+c_1\int_0^h |u(x', x_n)|^p dx_n.$$

To complete the proof, it is sufficient to integrate the latter inequality over Γ, then apply consequently Hölder inequality and Young inequality ($ab \le a^p/p + b^{p'}/p'$). □

Let us define a linear operator $\gamma : C^1(\overline{\Omega}) \subset W^{1,p}(\Omega) \to L^p(\partial\Omega)$ so that $\gamma u = u|_{\partial\Omega}$. By (3.2.1), this operator can be considered as a linear bounded operator from $W^{1,p}(\Omega)$ into $L^p(\partial\Omega)$ with domain of definition $D(\gamma) = C^1(\overline{\Omega})$, which is dense in $W^{1,p}(\Omega)$. Its operator norm

$$\|\gamma\|_{D(\gamma)} = \sup\{\|\gamma u\|_{p,\partial\Omega} : \ u \in D(\gamma), \|u\|_{p,1,\Omega} \le 1\}$$

is finite. It is known from the Functional Analysis that such an operator admits unique continuation $\widetilde{\gamma}$ to the whole $W^{1,p}(\Omega)$. The operator $\widetilde{\gamma} : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ has the following important properties:

(i) $\|\widetilde{\gamma}\| = \|\gamma\|_{D(\gamma)} \leq c(n,p,\Omega)$
(ii) $\widetilde{\gamma}u = \gamma u$ for all $u \in C^1(\overline{\Omega})$.

$\widetilde{\gamma}$ is the *trace* operator, acting on $W^{1,p}(\Omega)$ into $L^p(\partial\Omega)$, and $\widetilde{\gamma}u$ is the trace of a function $u \in W^{1,p}(\Omega)$. For the trace of u, we use the classical notation, i.e., $\widetilde{\gamma}u = u|_{\partial\Omega}$. It should be understood in the sense described above.

Lemma 2.4. *Let Ω be a bounded domain of class C^1 and let $1 < p < \infty$. Then we have*

$$\int\limits_{\Omega} uD_i v dx = \int\limits_{\partial\Omega} uv\nu_i dS - \int\limits_{\Omega} vD_i u dx \tag{3.2.3}$$

for all $u \in W^{1,p}(\Omega)$ and for all $v \in W^{1,p'}(\Omega)$ with $p' = \frac{p}{p-1}$ as usual. Here, ν is the unit outward normal to the surface $\partial\Omega$.

Proof (3.2.3) is valid for all u, v from $C^1(\overline{\Omega})$. Therefore, we can write (3.2.3) for sequences of smooth functions approximating functions $u \in W^{1,p}(\Omega)$ and $v \in W^{1,p'}(\Omega)$ and then take the limit using continuity of the trace operator with respect to strong convergence in Sobolev spaces. □

Corollary 2.5. *Let Ω be a bounded domain of class C^1 and $1 \leq p < \infty$. Suppose that $u \in W^{1,p}(\Omega)$ with $u|_{\partial\Omega} = 0$. Let $\Omega \subset \Omega_0$ and $\widetilde{u}$ is an extension of u by zero from Ω to Ω_0. Then $\widetilde{u} \in W^{1,p}(\Omega_0)$.*

Proof On problem sheet 3.

Remark 2.6. *Let $\widetilde{W}^{1,p}_0(\Omega) = \{u \in W^{1,p}(\Omega) : \ u|_{\partial\Omega} = 0\}$. In fact, $W^{1,p}_0(\Omega) \subseteq \widetilde{W}^{1,p}_0(\Omega)$ (explain why). However, under our assumptions on Ω (it is of class C^1), $W^{1,p}_0(\Omega) = \widetilde{W}^{1,p}_0(\Omega)$.*

Remark 2.7. *All above statements can be extended to bounded domains with Lipschitz boundaries.*

Chapter 4

Functional Methods for PDE's

4.1 Methods of Functional Analysis for PDE's

4.1.1 Notation

There are several ways to denote partial derivatives. Here, it is a list of some of them

$$\frac{\partial u}{\partial x_k} = D_k u = \partial_{x_k} u = u_{x_k} = u_{,k}.$$

In the rest of the course, the last notation will be used mostly both for classical and weak derivatives.

Another important thing is the so-called *nabla*-operator ∇. So, the action of this operator on u is the gradient of u and denoted by ∇u. This makes our notation closer to physical notation, in which fundamental equations of the physics are invariant (independent of a coordinate system). In particular, in Cartesian coordinates $x = (x_k)$, $\nabla u = (u_{,1}, u_{,2}, ..., u_{,n}) = (u_{,k})$. We also let $A : B = \text{sp} A^T B$ for two $n \times n$ matrices A and B. So that $A : B = A_{ij} B_{ij}$, where summation over repeated indices running from 1 to n is adopted. And for a given vector-valued field $a = (a_k)$, we denote $\operatorname{div} a = a_{k,k}$.

Now, we can consider two types of differential operators in Ω. The first one has a divergence form:

$$Lu := -\text{div}(a\nabla u) + b \cdot \nabla u + cu,$$

where a is a symmetric matrix-valued field, b is a vector-valued field, c is a scalar field in Ω. The second type has a non-divergence form:

$$Nu := -a : \nabla^2 u + b \cdot \nabla u + c.$$

In this course, we are going to deal with differential operators in the divergence form only.

Example

1. Laplace operator: here, $a = I := (\delta_{ij})$, $b = 0$, and $c = 0$. Then $-Lu = \mathrm{div}\nabla u = \triangle u = u_{,ii}$.
2. Helmholtz operator: $-Lu = \triangle u + k^2 u$

4.1.2 Dirichlet Boundary Value Problems for Elliptic Equation

We always assume that a symmetric matrix $a \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ satisfies the *ellipticity* condition

$$\nu|\xi|^2 \leq \xi \cdot a(x)\xi \leq \frac{1}{\nu}|\xi|^2, \qquad \forall \xi \in \mathbb{R}^n \tag{4.1.1}$$

a.e. in Ω with a positive constant ν.

Definition 1.1. *A function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying*

$$Lu = f + div\, g \qquad in\ \Omega \tag{4.1.2}$$

$$u = u_0 \qquad on\ \partial\Omega \tag{4.1.3}$$

with given f, g, and u_0 is called a classical solution to boundary value problem (4.1.2), (4.1.3).

Of course, a necessary condition for the existence of a classical solution to boundary value problem (4.1.2), (4.1.3) is sufficient smoothness of given functions a, b, c, f, g, and u_0. However, even under these conditions on the data of the problem, it is not so easy to prove the existence of classical solutions especially in the case of variable coefficients a, b, and c. On the other hand, there are quite a number of interesting and physically relevant cases, in which those coefficients are not smooth enough.

The modern way to tackle the solvability issue, which is, by the way, closely connected with modern ways to approximate solutions, is as follows. The classical set-up of boundary value problems is replaced with a weak setting based on integral identities rather than point-wise equations. This allows us to use powerful methods of functional analysis in order to prove existence theorems and develop a rigorous foundation of solving problems approximately, for example, by the finite-difference method or the finite element method.

In what follows, we always assume that

$$|a| := \sqrt{a : a} \in L^\infty(\Omega), \quad |b| := \sqrt{b \cdot b} \in L^\infty(\Omega), \quad c \in L^\infty(\Omega), \tag{4.1.4}$$

$$f \in L^2(\Omega), \qquad |g| \in L^2(\Omega), \tag{4.1.5}$$

$$u_0 \in H^1(\Omega) := W^{1,2}(\Omega), \tag{4.1.6}$$

and ellipticity condition (4.1.1) holds.

Definition 1.2. *A function $u \in H_0^1(\Omega) + u_0$ is a weak (or generalized) solution to boundary value problem (4.1.2), (4.1.3) if it satisfies the integral (variational) identity*

$$\mathcal{L}(u, w) = \int_\Omega (fw - g \cdot \nabla w)dx, \qquad w \in C_0^\infty(\Omega), \tag{4.1.7}$$

where $\mathcal{L}(u, w) := \int_\Omega ((a\nabla u) \cdot \nabla w + b \cdot \nabla uw + cuw)dx$.

Variational identity (4.1.7) is motivated by the following formal identity

$$\int_\Omega (Lu - f - \operatorname{div} g)wdx = 0, \qquad w \in C_0^\infty(\Omega),$$

which can be obtained by means of a single integration by parts involving the terms $w\mathrm{div}(a\nabla u)$ and $w\mathrm{div}\, g$.

Boundary condition (4.1.3) is satisfied in the sense of traces, i.e., $u - u_0 \in H_0^1(\Omega)$.

Remark 1.3. *If u is a weak solution, then variational identity (4.1.7) holds true for any test functions $w \in H_0^1(\Omega)$. (Explain why)*

Theorem 1.4. *(uniqueness implies existence) Let given functions a, b, and c satisfy conditions (4.1.1) and (4.1.4). Suppose, in addition, that any function $v \in H_0^1(\Omega)$ subject to the identity*

$$\mathcal{L}(v, w) = 0, \qquad \forall w \in C_0^\infty(\Omega),$$

must be equal to zero. Then, for any f, g, and u_0, satisfying conditions (4.1.5) and (4.1.6), boundary value problem (4.1.2), (4.1.3) has a unique weak solution.

Proof We are going to reduce the problem in question to Theorem 7.2. Our Hilbert space is going to be $U = H_0^1(\Omega)$. By Poincaré inequality, $|\cdot|_{2,1,\Omega}$ is a norm on U that is equivalent to the standard norm $\|\cdot\|_{2,1,\Omega}$, see Corollary 3.15. The ellipticity condition gives the following bounds

$$\sqrt{\nu}|u|_{2,1,\Omega} \leq \mathbf{I}u\mathbf{I}_{2,1,\Omega} := \Big(\int\limits_\Omega \nabla u \cdot a\nabla u\Big)^{\frac{1}{2}} \leq \frac{1}{\sqrt{\nu}}|u|_{2,1,\Omega}.$$

They imply that $\mathbf{I}\cdot\mathbf{I}_{2,1,\Omega}$ is a norm in U as well and it is equivalent to the norm $\|\cdot\|_{2,1,\Omega}$. The norm $\mathbf{I}\cdot\mathbf{I}_{2,1,\Omega}$ is generated by the scalar product

$$[u,v] = \int\limits_\Omega (a\nabla u)\cdot\nabla v\,dx = \int\limits_\Omega \nabla u\cdot a\nabla v\,dx, \qquad u,v\in U.$$

Then our bilinear form $\mathcal{L}$ can be presented as follows

$$\mathcal{L}(u,v) = [u,v] - \mathcal{L}_1(u,v),$$

where

$$\mathcal{L}_1(u,v) = -\int\limits_\Omega (vb\cdot\nabla u + cuv)dx, \qquad u,v\in U.$$

By Cauchy-Schwartz inequality, by Poincaré inequality, see Corollary 1.3, and by the above equivalence of norms, we

$$|\mathcal{L}_1(u,v)| \leq C_1(\|v\|_{2,\Omega}\|\nabla u\|_{2,\Omega} + \|v\|_{2,\Omega}\|u\|_{2,\Omega}) \leq C_2\mathbf{I}u\mathbf{I}_{2,1,\Omega}\mathbf{I}v\mathbf{I}_{2,1,\Omega}, \forall u,v\in U.$$

So, given $u\in U$, the linear functional $v\mapsto\mathcal{L}_1(u,v)$ is bounded in U. By Riesz theorem on representation of linear functional in Hilbert space, there exist a unique $K(u)\in U$ such that $\mathcal{L}_1(u,v) = [K(u),v]$ for any $u,v\in U$. It is easy to check that $K : U\to U$ is a bounded linear operator (indeed, $\mathbf{I}Ku\mathbf{I}_{2,1,\Omega} \leq C_2\mathbf{I}u\mathbf{I}_{2,1,\Omega}$). Our aim is to show that K is a compact operator. To this end, we should show that, for any bounded sequence u_m, the sequence Ku_m is precompact. WLOG, we may assume that $u_m \rightharpoonup u$ in U and thus $Ku_m \rightharpoonup Ku$ in U. If not, using a boundedness of u_m and a sufficient condition of weak compactness in Hilbert spaces, select a required subsequence, which could be denoted again by u_m. By Reillich-Kondrachov theorem, the embedding of U into $L^2(\Omega)$ is compact and we also may assume that $u_m\to u$ and $Ku_m\to Ku$ in $L^2(\Omega)$. Then, denoting $w_m = Ku_m$ and $w = Ku$, we show

$$\mathbf{I}K(u_m-u)\mathbf{I}^2_{2,1,\Omega} = [K(u_m-u), w_m - w] = \mathcal{L}_1(u_m-u, w_m-w) =$$

$$= \int_\Omega ((w_m - w)b \cdot \nabla(u_m - u) + c(w_m - w)(u_m - u))dx$$

$$\le C_1(\|w_m - w\|_{2,\Omega}\|\nabla u_m - \nabla u\|_{2,\Omega} + \|w_m - w\|_{2,\Omega}\|u_m - u\|_{2,\Omega}) \to 0$$

as $m \to \infty$. Indeed, a sequence $\|\nabla u_m - \nabla u\|_{2,\Omega}$ is bounded as $\nabla u_m - \nabla u \rightharpoonup 0$ in $L^2(\Omega)$ and $\|u_m - u\|_{2,\Omega} + \|w_m - w\|_{2,\Omega} \to 0$ by compact embedding mentioned above.

Finally, let us notice that

$$w \mapsto -\mathcal{L}(u_0, w) + \int_\Omega (fw - g \cdot \nabla w)dx$$

is a linear bounded functional in U (explain why) and thus, by Riesz theorem, there exists $F \in U$ such that

$$[F, w] = -\mathcal{L}(u_0, w) + \int_\Omega (fw - g \cdot \nabla w)dx, \quad \forall w \in U.$$

By our assumptions, the equation $w - Kw = 0$ ($\Leftrightarrow \mathcal{L}(u, v) = [u - Ku, v]$ $u, v \in U$) has the only trivial solution and, hence, by Fredholm Alternative, there exists a unique $\overline{u} \in U$ such that $\overline{u} - K\overline{u} = F$. According to our notation this is equivalent to the following identity

$$[\overline{u}, w] - [K\overline{u}, w] = [F, w], \qquad \forall w \in U$$

or

$$\mathcal{L}(\overline{u}, w) = -\mathcal{L}(u_0, w) + \int_\Omega (fw - g \cdot \nabla w)dx, \qquad \forall w \in U.$$

This means that $u = u_0 + \overline{u}$ is a required unique weak solution to boundary value problem (4.1.2), (4.1.3). □

Corollary 1.5. *Assume that our bilinear form $\mathcal{L}$ is <u>coercive</u> in the following sense: there exists a positive constant C such that*

$$\mathcal{L}(w, w) \ge C\|w\|^2_{2,\Omega}, \qquad \forall w \in C_0^\infty(\Omega). \tag{4.1.8}$$

Then boundary value problem (4.1.2), (4.1.3) has a unique weak solution.

Indeed, coercivity condition holds for all $w \in H^1_0(\Omega)$ (explain why). Now, let $v \in H^1_0(\Omega)$ satisfy the identity $\mathcal{L}(v, w) = 0$ for all $w \in C^\infty_0(\Omega)$ and thus for all $w \in H^1_0(\Omega)$. If we take a test function $w = v$, coercivity condition (4.1.8) implies $\|v\|_{2,\Omega} = 0$ and, hence, $v = 0$. Now, the statement of the corollary follows from the above theorem. □

EXAMPLE Let $\operatorname{div} b \leq 0$ and $c \geq 0$. We need to spell out how we understand $\operatorname{div} b \leq 0$ with $b = (b_i) \in L^1_{\mathrm{loc}}(\Omega)$. By definition, a distribution $T \geq 0$ if and only if $T(\varphi) \geq 0$ for any $\varphi \in C^\infty_0(\Omega)$ and $\varphi \geq 0$. So, $\operatorname{div} b \leq 0$ means that $\operatorname{div} T_b \leq 0$, which in turn means that $\operatorname{div} T_b(\varphi) = -T_b(\nabla\varphi) \leq 0$ for any non-negative $\varphi \in C^\infty_0(\Omega)$. Hence,

$$\int\limits_\Omega wb \cdot \nabla w dx = \int\limits_\Omega wb_i w_{,i} dx = \frac{1}{2}\int\limits_\Omega b_i (w^2)_{,i} dx = \frac{1}{2} T_b(\nabla|w|^2) \geq 0$$

for any $w \in C^\infty_0(\Omega)$. Then we find with the help of Poincaré inequality that

$$\mathcal{L}(w, w) = \int\limits_\Omega (\nabla w \cdot a\nabla w + wb \cdot \nabla w + cw^2) dx \geq \nu\|\nabla w\|^2_{2,\Omega} \geq C\|w\|^2_{2,\Omega}$$

for any $w \in C^\infty_0(\Omega)$.

It is interesting what happens if the main assumption of Theorem 1.4 is violated.

Theorem 1.6. *Assume that there exists $v_0 \in H^1_0(\Omega)$ such that v_0 is not identically zero and*

$$\mathcal{L}(v_0, w) = 0$$

for all $w \in C^\infty_0(\Omega)$. Then boundary value problem (4.1.2), (4.1.3) has a weak solution provided f, g, and u_0 satisfy compatibility conditions (4.1.5) and (4.1.6), and

$$\mathcal{L}(u_0, v) = \int\limits_\Omega (fv - g \cdot \nabla v) dx$$

for any $v \in H^1_0(\Omega)$ having the property

$$\mathcal{L}(w, v) = 0 \qquad \forall w \in C^\infty_0(\Omega).$$

Remark 1.7. *The identity $\mathcal{L}(w, v) = 0$ for $\forall w \in C^\infty_0(\Omega)$ is a weak form of the following homogeneous boundary value problem*

$$-\operatorname{div}(a\nabla v) - \operatorname{div}(bv) + cv = 0$$

in Ω and $v = 0$ on the boundary $\partial\Omega$.

PROOF The statement follows from a remark to Fredholm Alternative and the following identity

$$\mathcal{L}_1(v, w) = [Kv, w] = [v, K^* w]$$

being valid for any v and w in $H_0^1(\Omega)$. □

4.1.3 Variational Method

In some case, the existence of a weak solution to boundary value problems can be proved as a result of a variational approach. Let us consider the simplest case

$$-\mathrm{div}(a\nabla u) = f \tag{4.1.9}$$

in Ω and

$$u = 0 \tag{4.1.10}$$

on $\partial\Omega$.

We start with the following abstract version of the Weierstrass theorem.

Theorem 1.8. *Let V be a reflexive Banach space. Assume that we are given a functional $I : V \to]-\infty, \infty]$, having the following properties:*

(i) sequentially weak lower semi-continuity: for any sequence v^m such that $v^m \rightharpoonup v$, the following holds

$$\liminf_{m\to\infty} I(v^m) \geq I(v),$$

(ii) coercivity: if $\|v^m\|_V \to \infty$, then $I(v^m) \to +\infty$.

Suppose, further, that $I(v_1) < +\infty$ for some $v_1 \in V$. Then, there exists $u \in V$ such that

$$I(u) = A := \inf_{v\in V} I(v) > -\infty.$$

PROOF Let v^m be a minimising sequence, i.e., $I(v^m) \to A$. According to our assumptions, $A < +\infty$. By the coercivity condition, $\|v^m\|_V$ must be bounded. Since V is reflexive, there exists a subsequence still denoted by v^m such that $v^m \rightharpoonup u \in V$. By weak lower semi-continuity,

$$A = \lim_{m\to\infty} I(v^m) \geq I(u) > -\infty.$$

So, $I(u) = A$. □

The question to be raised is how one can check sequential weak lower semi-continuity.

Lemma 1.9. *Let V be a Banach space. Let I be strongly lower semi-continuous, i.e., $v^m \to v$ in V implies $\liminf_{m\to\infty} I(v^m) \geq I(v)$ and let I be convex, i.e.,*

$$I(\lambda u + (1-\lambda)v) \leq \lambda I(u) + (1-\lambda)I(v)$$

for all $u, v \in V$ and for all $0 \leq \lambda \leq 1$. Then I is a sequentially weak lower semi-continuous functional.

PROOF We are going to prove the lemma if $V = H$ is a Hilbert space. So assume that $v^m \rightharpoonup v$ in H. Without loss of generality, we may assume that

$$\lim_{m\to\infty} I(v^m) = \liminf_{m\to\infty} I(v^m).$$

By Banach-Sacks theorem, there exists a subsequence still denoted by v^m such that

$$u^m = \frac{1}{m}\sum_{k=1}^{m} v^k \to v$$

in V. By convexity, we have

$$I(u^m) \leq \frac{1}{m}\sum_{k=1}^{m} I(v^k) \to \lim_{m\to\infty} I(v^m).$$

It remains to notice that

$$\lim_{m\to\infty} I(u^m) \geq I(v).$$

□

Now, let us consider a functional

$$I(v) := \frac{1}{2}\int\limits_{\Omega} (a\nabla v)\cdot\nabla v dx - \int\limits_{\Omega} f v dx$$

for $V = H^1_0(\Omega)$. It is assumed that a satisfies the ellipticity condition (4.1.1) and $f \in L^2(\Omega)$. It is easy to check that the ellipticity condition provides convexity of our functional. Moreover, since it is continuous in V, the functional I is sequentially weakly lower semi-continuous on V. Moreover, the ellipticity condition

$$I(v) \geq \frac{\nu}{2}\|\nabla v\|^2_{2,\Omega} - \|f\|_{2,\Omega}\|v\|_{2,\Omega}.$$

Clearly, Poincare's inequality implies the coercivity condition. So, by Theorem 1.8, there exists $u \in V$ such that

$$I(u) = A := \inf_{v \in V} I(v).$$

Our aim is to show that u is a weak solution to (4.1.9) and (4.1.10). To this end, given $w \in C_0^\infty(\Omega)$ and $t > 0$, we derive from the last identity the following

$$0 \le I(u+tw) - I(u) = t\int\limits_\Omega ((a\nabla u)\cdot\nabla w - fw)dx + \frac{t^2}{2}\int\limits_\Omega (a\nabla w)\cdot\nabla w dx.$$

Dividing the latter inequality by t and tending $t \to 0$, we get

$$0 \le \int\limits_\Omega ((a\nabla u)\cdot\nabla w - fw)dx$$

for any $w \in C_0^\infty(\Omega)$. This certainly implies that u is a weak solution to (4.1.9) and (4.1.10).

4.1.4 Spectrum of Elliptic Differential Operators under Dirichlet Boundary Condition

Let us go back to our elliptic differential equations

$$Lu = -\mathrm{div}(a\nabla u) + b\cdot\nabla u + cu$$

with bounded coefficients $a = (a_{ij})$, $b = (b_i)$, and c in a bounded domain $\Omega \subset \mathbb{R}^3$, where the matrix-valued function satisfies the standard ellipticity condition. These are our standing assumptions. The corresponding bilinear form is

$$\mathcal{L}(u,v) = \int\limits_\Omega (\nabla v\cdot a\nabla u + b\cdot\nabla uv + cuv)dx$$

for any $u, v \in H^1(\Omega)$.

Now, we restrict ourselves to the case homogeneous Dirichlet boundary conditions. Formally, the latter means that we consider the above bilinear form on $H_0^1(\Omega)\times H_0^1(\Omega)$.

We shall say that the bilinear form $\mathcal{L}$ is *symmetric* in $H_0^1(\Omega) \times H_0^1(\Omega)$ if and only if

$$\mathcal{L}(u, v) = \mathcal{L}(v, u)$$

for any $u, v \in H_0^1(\Omega)$. Assume for a moment that a, b, and c are smooth. Then integration by parts gives:

$$\mathcal{L}(u, v) = \int_\Omega (Lu)v dx = \int_\Omega u(L^* v) dx$$

for any $u, v \in C_0^\infty(\Omega)$, where

$$L^* v = -\mathrm{div}(a\nabla v) - \mathrm{div}(bv) + cv$$

and L^* is called formally adjoint operator.

We can easily see that the bilinear form $\mathcal{L}$ is symmetric if and only if $L = L^*$. Indeed, we have in the sense of distributions

$$\mathcal{L}(u, v) = \int_\Omega (Lu)v dx = \int_\Omega u(L^* v) dx = \mathcal{L}(v, u) =$$

$$= \int_\Omega (Lv)u dx = \int_\Omega v(L^* u) dx = \int_\Omega (L^* u)v dx$$

for any $u, v \in C_0^\infty(\Omega)$.

Obviously, the condition of symmetry holds if

$$b = 0.$$

From now on we assume that the bilinear form $\mathcal{L}$ is symmetric on $U \times U$, where $U = H_0^1(\Omega)$, and that $c \geq 0$ in Ω. We know that for any $f \in L^2(\Omega)$, there exists a unique element $u \in U$ such that

$$\mathcal{L}(u, v) = (f, v) := \int_\Omega f v dx, \qquad \forall v \in U.$$

So, we have a well-defined operator $K : L^2(\Omega) \to L^2(\Omega)$ defined by $u = Kf$. Obviously, it is a compact operator in $L^2(\Omega)$ (explain why). Let us show that it is a symmetric operator. Indeed, for $u = Kf$ and $v = Kg$, we have

$$\mathcal{L}(u, w) = (f, w), \qquad \mathcal{L}(v, w) = (g, w)$$

for any $w \in U$. Hence,

$$(Kf, g) = (u, g) = (g, u) = \mathcal{L}(v, u) = \mathcal{L}(u, v) =$$

$$= (f, v) = (v, f) = (Kg, f) = (f, Kg)$$

for any $f, g \in L^2(\Omega)$. Symmetry follows.

Now, we can apply the celebrated Hilbert-Schmidt theorem about spectrum of a symmetric compact operator. It reads the following.

Theorem 1.10. *Let H be a Hilbert space and $K : H \to H$ be a symmetric compact operator. There exists an orthonormal system $\{\varphi_m\}_{m=1}^N \subset H$ that consists of eigenfunctions φ_m belonging to eigenvalue $\mu_m \neq 0$, i.e., $K\varphi_m = \mu_m \varphi_m$, such that for any $h \in H$ one has a unique representation*

$$h = \sum_{m=1}^{N} c_m \varphi_m + h'$$

with $c_m = (h, \varphi_m)$ and $Kh' = 0$.

Moreover, if $N = \infty$, then $\mu_m \to 0$ as $m \to \infty$.

Let us discuss consequences of the Hilbert-Schmidt theorem for our particular case with $H = L^2(\Omega)$. First of all, it is easy to see that the equation $Kh = 0$ has the only trivial solution $h = 0$, i.e., $\mu = 0$ is not an eigenvalue of the operator K. Hence, $N = \infty$ and $\{\varphi_m\}_{m=1}^{\infty}$ is an orthogonal basis in $L^2(\Omega)$. We let $\lambda_m = 1/\mu_m$. Then the identity $K\varphi_m = \mu_m \varphi_m$ is equivalent to

$$\mathcal{L}(\varphi_m, v) = \lambda_m (\varphi_m, v)$$

for any $v \in U$. Here, of course, $\|\varphi_m\|_{2,\Omega} = 1$. By ellipticity conditions,

$$\mathcal{L}(\varphi_m, \varphi_m) = \lambda_m > 0.$$

One can numerate eigenvalues in the following way

$$0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_m \leq ...$$

with $\lambda_m \to \infty$. Here, a particular eigenvalue is repeated as many times as its multiplicity that is the dimension of the subspace

$$\{u \in U : \ \mathcal{L}(u, v) = \lambda(u, v) \quad \forall v \in U\}.$$

It is known that this dimension is finite.

As it follows from the Hilbert-Schmidt theorem, for any $h \in L^2(\Omega)$,

$$h = \sum_{m=1}^{\infty} c_m \varphi_m, \qquad c_m = (h, \varphi_m)$$

and the series converges in $L^2(\Omega)$.

Now, our aim is to show if $h \in U$ the above series converges in U as well. To this end, let us introduce a scalar product as follows

$$[u, v] := \mathcal{L}(u, v), \quad u, v \in U.$$

We know that

$$\|h - \sum_{m=1}^{N} c_m \varphi_m\|_{2,\Omega} \to 0$$

as $N \to \infty$. By definition of eigenvalues,

$$[\varphi_m, h] = \lambda_m(\varphi_m, h) = \lambda_m c_m.$$

Therefore,

$$[h, \sum_{m=1}^{N} c_m \varphi_m] = \sum_{m=1}^{N} \lambda_m c_m^2. \tag{4.1.11}$$

On the other hand, we can find

$$\mathbf{I}h - \sum_{m=1}^{N} c_m \varphi_m \mathbf{I}_{2,1,\Omega}^2 = [h - \sum_{m=1}^{N} c_m \varphi_m, h - \sum_{m=1}^{N} c_m \varphi_m] =$$

$$= \mathbf{I}h\mathbf{I}_{2,1,\Omega}^2 - \sum_{m=1}^{N} \lambda_m c_m^2 \geq 0.$$

Therefore, we can state that series

$$\sum_{m=1}^{\infty} \lambda_m c_m^2 \tag{4.1.12}$$

converges and moreover

$$\mathbf{I} \sum_{m=1}^{N} c_m \varphi_m \mathbf{I}_{2,1,\Omega}^2 = \sum_{m=1}^{N} \lambda_m c_m^2 \leq \sum_{m=1}^{\infty} \lambda_m c_m^2 \leq \mathbf{I}h\mathbf{I}_{2,1,\Omega}^2 < \infty.$$

Now, let $g_N = \sum_{m=1}^{N} c_m \varphi_m$. There exists a subsequence such that g_{N_k} converges weakly to g in U. Then from (4.1.11) and (4.1.12), it follows that

$$[h, g] = \sum_{m=1}^{\infty} \lambda_m c_m^2.$$

But $\sum_{m=1}^{N} c_m \varphi_m \to h$ in $L^2(\Omega)$ as $N \to \infty$ and thus $g = h$. Hence,

$$\sum_{m=1}^{\infty} \lambda_m c_m^2 = \mathbf{I} h \mathbf{I}_{2,1,\Omega}^2$$

and thus $g_N \to h$ in U. That is all.

4.2 Smoothness of Weak Solutions

4.2.1 The Second Embedding Theorem

In what follows, we shall use the following notion for mean values: $[u]_\Omega := \frac{1}{|\Omega|} \int_\Omega u dx$.

We start with a technical lemma that shows how we can approximate a function at a point by mean values.

Lemma 2.1. *Let $u \in L^1(B(x_0, R))$. Assume that there exist positive constant A and α such that, for all $0 < r \leq R$,*

$$\Psi(x_0, r) := \frac{1}{|B(r)|} \int_{B(x_0,r)} |u - [u]_{B(x_0,r)}| dx \leq A r^\alpha. \tag{4.2.1}$$

Then there exists

$$\lim_{r \to 0} [u]_{B(x_0,r)} =: u_0$$

and, for all $0 < r \leq R$,

$$|u_0 - [u]_{B(x_0,r)}| \leq c A r^\alpha \tag{4.2.2}$$

with a constant c depending on n and α only.

Proof Let $r \le R$, then we have

$$|[u]_{B(x_0,r/2)} - [u]_{B(x_0,r)}| \le \frac{1}{|B(x_0,r/2)|} \int\limits_{B(x_0,r)} |u(x) - [u]_{B(x_0,r)}|dx \le$$

$$\le 2^n \Psi(x_0, r).$$

Thus, for $m > k \ge 0$,

$$|[u]_{B(x_0,r/2^m)} - [u]_{B(x_0,r/2^k)}| \le \sum_{i=k}^{m-1} |[u]_{B(x_0,r/2^{i+1})} - [u]_{B(x_0,r/2^i)}|$$

$$\le 2^n \sum_{i=k}^{m-1} \Psi(r/2^i) \le 2^n A \sum_{i=k}^{m-1} (r/2^i)^\alpha = 2^n r A \sum_{i=k}^{m-1} (1/2^i)^\alpha \to 0 \tag{4.2.3}$$

as $k \to \infty$. So, $\lim\limits_{m\to\infty} [u]_{B(x_0,r/2^m)}$ exists for any $0 < r \le R$. Let $u_0 := \lim\limits_{m\to\infty} [u]_{B(x_0,R/2^m)}$. Our aim is to show that $\lim\limits_{r\to 0} [u]_{B(x_0,r)} = u_0$. Indeed, given $r_k \to 0$, we can find subsequence of m_k such that

$$R/2^{m_k+1} \le r_k \le R/2^{m_k}$$

for any k. Then we can repeat the above arguments to show that

$$|[u]_{B(x_0,r_k)} - [u]_{B(x_0,R/2^{m_k+1})}| \le 2^n \Psi(x_0, r_k) \le 2^n A r_k^\alpha \to 0.$$

So, $[u]_{B(x_0,r_k)} \to u_0$.

To derive (4.2.2) from (4.2.3), it is sufficient to let $k = 0$ there and then pass to the limit as $m \to \infty$. □

Now, assuming that Ω is bounded, we introduce Hölder space $C^\alpha(\bar\Omega)$, which consists of all continuous functions $f : \bar\Omega \to \mathbb{R}$ such that

$$\|u\|_{C^\alpha(\bar\Omega)} := \|u\|_{\infty,\Omega} + [u]_{\alpha,\Omega} < \infty,$$

where

$$[u]_{\alpha,\Omega} := \sup_{x,y\in\Omega, x\ne y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

$C^\alpha(\bar\Omega)$ is a Banach space, see Problem Sheet 4.

Lemma 2.2. *(Campanato) Let* $u \in L^1(B(R))$. *Assume that there exist positive constant* A *and* α *such that*

$$\Psi(x_0,r) := \frac{1}{|B(r)|}\int\limits_{B(x_0,r)} |u-[u]_{B(x_0,r)}|dx \leq Ar^\alpha \tag{4.2.4}$$

for all $B(x_0,r)\subset B(R)$. *Then, for any* $0<\varrho<R$,

$$\|u\|_{C^\alpha(\bar{B}(\varrho))} \leq C(n,\alpha,\varrho,R)(A+\|u\|_{1,B(R)}). \tag{4.2.5}$$

Proof First of all, by Lemma 2.1, $\lim\limits_{r\to 0}[u]_{B(x,r)}$ exists for all $x\in B(R)$. The function $x\mapsto \lim\limits_{r\to 0}[u]_{B(x,r)}$ belongs to the equivalence class u and in what follows we shall work with this particular representative which is going to be denoted simply by u.

Now, we let us fix a positive number $\varrho<R$. Letting $r=(R-\varrho)/2$, we deduce form Lemma 2.1 that for all $x\in B(\varrho)$

$$|u(x)|\leq cAr^\alpha + |[u]_{B(x,r)}| \leq C(R-\varrho,n,\alpha)(A+\|u\|_{1,B(R)}). \tag{4.2.6}$$

To proceed further, we assume that x_0 and y_0 belong to the ball $B(\varrho)$ and $|x_0-y_0|<R-\varrho$. Let $z_0=(x_0+y_0)/2$ and $2r=|x_0-y_0|>0$. Then

$$|u(x_0)-u(y_0)|\leq |u(x_0)-[u]_{B(x_0,r)}|+|u(y_0)-[u]_{B(y_0,r)}|+$$

$$+|[u]_{B(x_0,r)}-[u]_{B(z_0,2r)}|+|[u]_{B(y_0,r)}-[u]_{B(z_0,2r)}|.$$

The first two terms on the right hand side can be estimated with the help of (4.2.2). Both them give the right contribution. The third and fourth terms are estimated in the same way. Let us treat the third one. So, we have, by Corollary 2.4,

$$|[u]_{B(x_0,r)}-[u]_{B(z_0,2r)}|\leq \frac{1}{|B(x_0,R)|}\int\limits_{B(x_0,r)}|u(x)-[u]_{B(z_0,2r)}|dx\leq$$

$$\leq \frac{1}{|B(x_0,r)|}\int\limits_{B(z_0,2r)}|u(x)-[u]_{B(z_0,2r)}|dx\leq$$

$$\leq \frac{|B(z_0,2r)|}{|B(x_0,r)|}\Psi(z_0,2r)\leq c(n,\alpha)Ar^\alpha \leq c(n,\alpha)A|x_0-y_0|^\alpha.$$

So,

$$|u(x_0)-u(y_0)|\le c(n,\alpha)A|x_0-y_0|^\alpha$$

provided x_0 and y_0 belong to the ball $B(\varrho)$ and $|x_0-y_0|<R-\varrho$. The last restriction can be easily removed with the help of estimate (4.2.6). Indeed, let x_0 and y_0 are in $B(\varrho)$ but $|x_0-y_0|\ge R-\varrho$. Then

$$|u(x_0)-u(y_0)|\le 2\|u\|_{\infty,B(\varrho)}\le 2\Big(\frac{|x_0-y_0|}{R-\varrho}\Big)^\alpha\|u\|_{\infty,B(\varrho)}.$$

Summarizing the above estimates, we arrive at (4.2.5). □

Lemma 2.3. *(Poincaré-Sobolev) Let Ω be a bounded Lipschitz domain. Then*

$$\|u-[u]_\Omega\|_{p,\Omega}\le c(n,p,\Omega)|u|_{p,1,\Omega},\qquad \forall u\in W^{1,p}(\Omega).\tag{4.2.7}$$

Here, $[u]_\Omega:=\frac{1}{|\Omega|}\int\limits_\Omega u dx.$

Proof Our proof is based on the Reillich-Kondrachov theorem. Suppose that the statement is false. Then for any $m\in\mathbb{N}$ there exists $u_m\in W^{1,p}(\Omega)$ such that $\|u_m-[u_m]_\Omega\|_{p,\Omega}>m|u_m-[u_m]_\Omega|_{p,1,\Omega}$. Letting $v_m:=(u_m-[u_m]_\Omega)/\|u_m-[u_m]_\Omega\|_{p,\Omega}$, we have

$$\|v_m\|_{p,\Omega}=1>m|v_m|_{p,1,\Omega},\qquad [v_m]_\Omega=0.\tag{4.2.8}$$

From (4.2.8), it follows that v_m is bounded in $W^{1,p}(\Omega)$ and, by Reillich-Kondrachov theorem, sequence v_m is precompact in $L^p(\Omega)$. Hence, there exists a subsequence $v_{m_k}\to v$ in $L^p(\Omega)$. The limit function v must have the vanishing mean value, i.e., $[v]_\Omega=0$ and satisfy the identity $\|v\|_{p,\Omega}=1$. In addition, from (4.2.8), we deduce that $D_iv_{m_k}\to 0$ in $L^p(\Omega)$ for all $i=1,2,...,n$. Therefore, v has all the weak derivatives that are equal to zero. So, v is a constant in Ω. This constant must be equal to zero since v has zero mean value in Ω. But this is in a contradiction with $\|v\|_{p,\Omega}=1$. □

Corollary 2.4. *If $\Omega=B(x_0,R)$, then*

$$\|u-[u]_{B(x_0,R)}\|_{p,B(x_0,R)}\le c(n,p)R|u|_{p,1,B(x_0,R)},\qquad \forall u\in W^{1,p}(B(x_0,R)).$$

The corollary is proved by scaling. Change variables so that $v(y)=u(x)$ provided $y=(x-x_0)/R$, where $y\in B(0,1)$. Then we use Lemma 2.3 for $\Omega=B(0,1)$ and return to the old coordinates. □

Theorem 2.5. *Let Ω be a bounded domain with Lipschitz boundary and let $n < p < \infty$. Then the Sobolev space $W^{1,p}(\Omega)$ is continuously embedded into space $C^\beta(\overline{\Omega})$ for any $0 \leq \beta \leq \alpha = 1 - n/p$.*

Moreover, the embedding is compact if $0 \leq \beta < \alpha$.

PROOF Our first remark is follows. It is sufficient to prove continuity of embedding $W^{1,p}(\Omega)$ into $C^\alpha(\overline{\Omega})$. Indeed, continuity of others embeddings follows from the fact $C^\beta(\overline{\Omega}) \subset C^\alpha(\overline{\Omega})$ if $\beta \leq \alpha$ (Explain why). The statement about compactness can be deduced from the fact that $C^\beta(\overline{\Omega})$ is embedded into $C^\alpha(\overline{\Omega})$ if $\beta < \alpha$, see Problem Sheet 4.

Let us take a number $R > 0$ so large that $\Omega \Subset B(R/2)$ and fix it. Let $v \in W^{1,p}_0(B(R/2))$ be an extension of a given function $u \in W^{1,p}(\Omega)$ with the following estimate

$$\|v\|_{p,1,B(R/2)} \leq c(\Omega, R, n, p)\|u\|_{p,1,\Omega}. \tag{4.2.9}$$

The function v can be extended to the whole ball $B(R)$ by zero. This extension is still a function from $W^{1,p}_0(B(R))$ and equal to zero in $B(R) \setminus B(R/2)$. We have from the Corollary 2.4 and from Hölder inequality the following estimate

$$\frac{1}{|B(r)|}\int\limits_{B(x_0,r)} |v - [v]_{B(x_0,r)}|dx \leq$$

$$\leq \Big(\frac{1}{|B(r)|}\int\limits_{B(x_0,r)} |v - [v]_{B(x_0,r)}|^p dx\Big)^{\frac{1}{p}} \leq c(n,p) r^\alpha |v|_{p,1,B(x_0,R)}$$

for $B(x_0, r) \subset B(R)$. Then, by Lemma 2.2, $v \in C^\alpha(\overline{B}(R/2))$ with estimate

$$\|v\|_{C^\alpha(\overline{B}(R/2))} \leq c(\Omega, R, n, p)(|v|_{p,1,B(x_0,R)} + \|v\|_{1,B(R)}).$$

It remains to notice that

$$\|u\|_{C^\alpha(\overline{\Omega})} \leq \|v\|_{C^\alpha(\overline{B}(R/2))}$$

and by Hölder inequality and by (4.2.9)

$$|v|_{p,1,B(x_0,R)} + \|v\|_{1,B(R)} \leq c(\Omega, R, n, p)\|u\|_{p,1,\Omega}.$$

This completes the proof of the theorem. $\square$

4.2.2 Solvability in $H^2 \cap H_0^1$

Let us consider the following elliptic differential operator of the form

$$Lu = -\mathrm{div}(a\nabla u) + b \cdot \nabla u + cu$$

with coefficients satisfying our standing assumptions in a bounded domain Ω with sufficiently smooth boundary, for example, of class C^2. We assume in addition that a is continuously differentiable in $\overline{\Omega}$. Then

$$|\nabla a| \leq \mu < \infty$$

in $\overline{\Omega}$ for some μ. Our goal is to show that, for any $f \in L^2(\Omega)$, the Dirichlet boundary value problem

$$Lu = f$$

in Ω,

$$u|_{\partial\Omega} = 0$$

has a solution u that belongs to $H^2(\Omega)$.

We first notice that

$$Lu = -a_{ij}u_{,ij} + (b_j - a_{ij,i})u_{,j} + cu$$

and then by our assumptions the operator L is bounded on $H^2(\Omega)$, i.e.,

$$\|Lu\|_{2,\Omega} \leq c\|u\|_{2,2,\Omega}$$

for all $u \in H^2(\Omega)$.

We also introduce the space

$$\mathcal{E}(\Omega) = \{v \in C^2(\overline{\Omega}) : \ v|_{\partial\Omega} = 0\}$$

and its closure in $H^2(\Omega)$, i.e.,

$$H_+^2(\Omega) = [\mathcal{E}(\Omega)]^{H^2(\Omega)}.$$

Clearly, $H_+^2(\Omega)$ is a subspace of $H_0^1(\Omega) \cap H^2(\Omega)$ (exercise). Now, let us consider the restriction of the operator L on $H_+^2(\Omega)$. We denote it by the same symbol L. Our aim is to show that $R(L) := L(H_+^2(\Omega)) = L^2(\Omega)$. This would be an answer to the question of solvability of our Dirichlet boundary value problem in $H_0^1 \cap H^2$.

We start with an important auxiliary statement.

Proposition 2.6. *There exists a constant C depending on Ω, ν, μ, $\|b\|_{\infty,\Omega}$, and $\|c\|_{\infty,\Omega}$ such that*

$$\|\nabla^2 u\|^2_{2,\Omega} \leq C(\|Lu\|^2_{2,\Omega} + \|u\|^2_{2,\Omega})$$

for any $u \in H^2_+(\Omega)$.

Proof. By definition of the space $H^2_+(\Omega)$, it is sufficient to prove the estimate of the proposition for functions $u \in \mathcal{E}(\Omega)$ only.

Just to understand the main idea better, we prove the proposition in the the simplest case $Lu = -\Delta u$. Let us fix Cartesian coordinates $x = (x_i)$ in $\mathbb{R}^n$. After integration by parts, we have

$$\int\limits_\Omega |\nabla^2 u|^2 dx = \int\limits_\Omega u_{,ij}u_{,ij}dx = -\int\limits_{\partial\Omega} (\nu_j u_{,j}u_{,ii} - u_{,ij}u_{,j}\nu_i)ds + \int\limits_\Omega u_{,ii}u_{,jj}dx =$$

$$= -\int\limits_{\partial\Omega} Ids + \int\limits_\Omega u_{,ii}u_{,jj}dx,$$

where

$$I := \nu \cdot \nabla u \Delta u - \nu \otimes \nabla u : \nabla^2 u$$

and ν is the unit outward normal to the surface $\partial\Omega$.

Let x_0 be an arbitrary point on the boundary $\partial\Omega$. We also can find a local Cartesian coordinates y centred at the point x_0 so that the axis y_n has the same direction as the unit outward normal ν to $\partial\Omega$ at the point x_0. So, we have $y = Q^T(x - x_0)$, where $Q = (c_{kl})$ is an orthogonal matrix and Q^T is transpose of it. Then we let

$$v(y) := u(Qy + x_0)$$

and the change of variables gives to us:

$$\frac{\partial u}{\partial x_k} = \frac{\partial v}{\partial y_s}c_{ks}, \quad \frac{\partial^2 u}{\partial x_k \partial x_l} = \frac{\partial^2 v}{\partial y_s \partial y_t}c_{lt}c_{ks}.$$

Now, since I is invariant with respect to shifts and rotations, we have

$$I(x_0) = \frac{\partial v}{\partial y_n}\frac{\partial^2 v}{\partial y_k \partial y_k} - \frac{\partial v}{\partial y_j}\frac{\partial^2 v}{\partial y_j \partial y_n}.$$

Adopting summation over repeated Greek indices running from 1 to $n-1$, we find

$$I(x_0) = \frac{\partial v}{\partial y_n}\frac{\partial^2 v}{\partial y_\alpha \partial y_\alpha} - \frac{\partial v}{\partial y_\beta}\frac{\partial^2 v}{\partial y_\beta \partial y_n}.$$

We may assume that the boundary $\partial\Omega$ in a neighbourhood of the point x_0 (or $y = 0$) is a graph of the function φ, i.e., $y_n = \varphi(y')$, where $y' = (y_1, y_2, ..., y_{n-1})$. By our construction, $\varphi(0) = 0$ and

$$\frac{\partial \varphi}{\partial y_\alpha}(0) = 0$$

with $\alpha = 1, 2, ..., n-1$.

We know that $h(y') = v(y', \varphi(y')) = 0$. Since

$$0 = \frac{\partial h}{\partial y_\alpha} = \frac{\partial v}{\partial y_\alpha} + \frac{\partial v}{\partial y_n}\frac{\partial \varphi}{\partial y_\alpha}.$$

It follows from the latter identity that

$$\frac{\partial v}{\partial y_\alpha}(0) = 0.$$

After further differentiations, we find

$$0 = \frac{\partial^2 h}{\partial y_\alpha \partial y_\beta} = \frac{\partial^2 v}{\partial y_\alpha \partial y_\beta} + \frac{\partial^2 v}{\partial y_\alpha \partial y_n}\frac{\partial \varphi}{\partial y_\beta} + \frac{\partial v}{\partial y_n}\frac{\partial^2 \varphi}{\partial y_\alpha \partial y_\beta} + \frac{\partial^2 v}{\partial y_n \partial y_\beta}\frac{\partial \varphi}{\partial y_\alpha} +$$

$$+ \frac{\partial^2 v}{\partial y_n^2}\frac{\partial \varphi}{\partial y_\beta}\frac{\partial \varphi}{\partial y_\alpha}$$

and, since

$$\frac{\partial u}{\partial \nu}(x_0) = \frac{\partial v}{\partial y_n}(0),$$

the following is true

$$\frac{\partial^2 v}{\partial y_\alpha \partial y_\alpha}(0) = -\frac{\partial u}{\partial \nu}(x_0)\frac{\partial^2 \varphi}{\partial y_\alpha \partial y_\alpha}(0).$$

Hence,

$$I(x_0) = -\Big|\frac{\partial u}{\partial \nu}(x_0)\Big|^2 \frac{\partial^2 \varphi}{\partial y_\alpha \partial y_\alpha}(0).$$

It is interesting to notice that if the domain Ω is convex then $\varphi_{,\alpha\alpha}(0) \leq 0$ for any point $x_0 \in \partial\Omega$ and, therefore, the following remarkable inequality is valid:

$$\|\nabla^2 u\|_{2,\Omega} \leq \|\Delta u\|_{2,\Omega}$$

for any $u \in \mathcal{E}(\Omega)$.

In general case, since the domain Ω is of class C^2, there exists a constant K independent of $x_0 \in \partial\Omega$ such that

$$\Big|\frac{\partial^2 \varphi}{\partial y_\alpha \partial y_\alpha}(0)\Big| \leq K.$$

So, we have the inequality

$$\int\limits_\Omega |\nabla^2 u|^2 dx \leq K \int\limits_{\partial\Omega} |\nabla u|^2 ds + \int\limits_\Omega |\Delta u|^2 dx.$$

By a simple modification of the proof of the theorem on traces, see Lemma 2.3, (explain what modification should be made) we have the following statement: given $\varepsilon > 0$, there exists a constant $C(\varepsilon, \Omega)$ such that

$$\int\limits_{\partial\Omega} |\nabla u|^2 ds \leq \varepsilon \int\limits_\Omega |\nabla^2 u|^2 dx + C(\varepsilon, \Omega) \int\limits_\Omega |\nabla u|^2 dx.$$

Picking up ε by the identity $K\varepsilon = 1/2$, we easily find

$$\frac{1}{2} \int\limits_\Omega |\nabla^2 u|^2 dx \leq C(K, \varepsilon, \Omega) \int\limits_\Omega |\nabla u|^2 dx + \int\limits_\Omega |\Delta u|^2 dx.$$

On the other hand, integration by parts gives

$$\int\limits_\Omega |\nabla u|^2 dx = -\int\limits_\Omega u \Delta u dx \leq \|u\|_{2,\Omega} \|\Delta u\|_{2,\Omega}.$$

The proposition follows. □

In what follows, we assume that the operator L satisfies the additional condition

$$(Lu, u) = \int\limits_\Omega u L u dx = \mathcal{L}(u, u) \geq \delta \|u\|^2_{2,\Omega}$$

for some positive δ, for any $u \in \mathcal{E}(\Omega)$ and thus for any $u \in H^2_+(\Omega)$. Then the inequality of Proposition 2.6 takes the form

$$\|\nabla^2 u\|_{2,\Omega} \le C\|Lu\|_{2,\Omega}$$

for any $u \in H^2_+(\Omega)$ with a constant C independent of u.

Our main theorem is as follows.

Theorem 2.7. *Assume that all above listed conditions on the operator L hold. Suppose that there exists an elliptic operator L_0 satisfying the same condition as the operator L but with possibly different parameters a_0, b_0, c_0, ν_0, μ_0, and δ_0. Assume that there exists a set $\mathcal{M} \subset R(L_0)$ that is dense in $L^2(\Omega)$.*

Then, for any $\tau \in [0,1]$, $R(L_\tau) = L^2(\Omega)$, where $L_\tau := L_0 + \tau(L - L_0) : H^2_+(\Omega) \to L^2(\Omega)$. Moreover, L_τ is injective and there exists a bounded inverse operator $L_\tau^{-1} : L^2(\Omega) \to H^2_+(\Omega)$.

Let us discuss simple applications of Theorem 2.7.

Let Ω be a ball in $\mathbb{R}^n$. It is well known that all eigenfunctions of the Laplace operator

$$-\Delta u = \lambda u$$

under the Dirichlet boundary conditions

$$u|_{\partial\Omega} = 0$$

are infinitely smooth. Since those eigenfunctions are dense in $L^2(\Omega)$, the operator $L_0 = -\Delta$ satisfies the assumptions of Theorem 2.7. Indeed, for any $f \in L^2(\Omega)$, we have

$$f = \sum_{m=1}^{\infty} c_m \varphi_m,$$

where $\{\varphi_m\}_{m=1}^{\infty}$ is an orthonormal basis in $L^2(\Omega)$ consisting of the eigenfunctions of the Laplace operator under the Dirichlet boundary condition. The solution of the problem

$$-\Delta u = \sum_{m=1}^{N} c_m \varphi_m, \qquad u|_{\partial\Omega} = 0$$

is

$$u = -\sum_{m=1}^{N} c_m \varphi_m / \lambda_m \in H^2_+(\Omega).$$

Similar arguments work if Ω can be transformed into the ball B by smooth non-degenerate change of variables $y = \psi(x)$ with $\psi \in C^3(\overline{\Omega})$. Let us show how to construct such an operator L_0 for the domain Ω. Take two functions $u, v \in H^2_+(\Omega)$ and let $u'(y) = u(x)$ and $v'(y) = v(x)$ for $y = \psi(x)$. We know that $u', v' \in H^2_+(B)$. Then simply by chain rule, we find

$$\int_B \frac{\partial u'}{\partial y_i}\frac{\partial v'}{\partial y_i} dy = \int_\Omega \frac{\partial u}{\partial x_k}\frac{\partial x_k}{\partial y_i}\frac{\partial v}{\partial x_s}\frac{\partial x_s}{\partial y_i} J dx, \tag{4.2.10}$$

where $J(x) = \det(\nabla\psi(x))$ is the Jacobian of the coordinate transformation $y = \psi(x)$ that satisfies the inequalities

$$0 < \alpha \le J(x) \le \beta < \infty$$

for all $x \in \Omega$. This identity suggests to introduce the operator L_0 as follows:

$$L_0 u(x) = -\frac{\partial}{\partial x_k}\Big(a_{ks}(x)\frac{\partial u}{\partial x_s}(x)\Big),$$

where $a(x) = g(x)J(x)$,

$$g_{ks}(x) = \frac{\partial x_k}{\partial y_i}(y)\frac{\partial x_s}{\partial y_i}(y)\Big|_{y=\psi(x)}.$$

Therefore, (4.2.10) implies

$$\int_B \frac{\partial u'}{\partial y_i}\frac{\partial v'}{\partial y_i} dy = \int_\Omega L_0(u) v dx.$$

To see that the matrix a satisfies the ellipticity condition, we let

$$d_+ = \sup_{y\in\overline{B}}\sup_{|\xi|=1} |(\nabla_y x(y))^T\xi|^2, \qquad d_- = \inf_{y\in\overline{B}}\inf_{|\xi|=1} |(\nabla_y x(y))^T\xi|^2.$$

It is easy to see $d_- > 0$ and $d_+ < \infty$ (explain why). We can pick up ν_0 sufficiently small so that $0 < \nu_0 \le \alpha d_-$ and $\beta d_+ \le \nu_0^{-1}$.

Then we find

$$\int_\Omega |u|^2 dx = \int_B |u'|^2 J^{-1} dy \le \frac{1}{\alpha}\int_B |u'|^2 dy \le \frac{1}{\lambda_1 \alpha}\int_B \frac{\partial u'}{\partial y_i}\frac{\partial u'}{\partial y_i} dy \le$$

$$\leq \frac{1}{\lambda_1 \alpha}(L_0 u, u).$$

Next, let $\varphi'_m(y)$ is an eigenfunction of the Laplace operator under the Dirichlet boundary condition in the ball B and let $\varphi_m(x) = \varphi'_m(y)$ provided $y = \psi(x)$. Then from (4.2.10) it follows that

$$\lambda_m \int_\Omega J\varphi_m v dx = \lambda_m \int_B \varphi'_m v' dy = \int_B \frac{\partial \varphi'_m}{\partial y_i}\frac{\partial v'}{\partial y_i} dy = -\int_\Omega L_0 \varphi_m v dx$$

and thus $L_0\varphi_m = \lambda_m J\varphi_m$ and $\varphi_m \in H^2_+(\Omega)$.

Now, take $f \in L^2(\Omega)$ and let $f'(y) = f(x)/J(x)$. Given $\varepsilon > 0$, we can find N such that

$$\int_B |f' - \sum_{m=1}^N c_k \varphi'_m|^2 dy < \varepsilon.$$

After change of variables, we find

$$\varepsilon > \int_\Omega |fJ^{-1} - \sum_{m=1}^N c_k \varphi_m|^2 J dx = \int_\Omega |f - \sum_{m=1}^N c_k J\varphi_m|^2 J^{-1} dx \geq$$

$$\geq \frac{1}{\beta}\int_\Omega |f - \sum_{m=1}^N c_k J\varphi_m|^2 dx.$$

So, image of the operator L_0 contains a set that is dense in $L^2(\Omega)$. Hence, if the operator L satisfies all the assumptions of Theorem 2.7 in such a domain Ω, then $R(L) = L^2(\Omega)$.

Proof of Theorem 2.7 Let us describe some properties of the operator L_τ. First of all,

$$\|L_\tau\| \leq \tau\|L\| + (1-\tau)\|L_0\| \leq \max\{\|L\|, \|L_0\|\} = c_3 \tag{4.2.11}$$

for any $\tau \in [0,1]$. Next,

$$(L_\tau u, u) = \tau(Lu, u) + (1-\tau)(L_0 u, u) \geq (\tau\delta + (1-\tau)\delta_0)\|u\|^2_{2,\Omega} \geq$$

$$\geq \delta_1 \|u\|^2_{2,\Omega},$$

where $\delta_1 = \min\{\delta, \delta_0\}$. The latter, together with Cauchy-Schwarz inequality, gives the estimate $\|L_\tau u\|_{2,\Omega} \geq \delta_1 \|u\|_{2,\Omega}$. Proposition 2.6 gives us

$$\|\nabla^2 u\|_{2,\Omega} \leq c'_3 \|L_\tau u\|_{2,\Omega}$$

with constant independent of τ. Moreover, by integration by parts,

$$\int_\Omega |\nabla u|^2 dx = -\int_\Omega \Delta u u dx \leq \|\nabla^2 u\|_{2,\Omega}\|u\|_{2,\Omega} \leq c_3'/\delta_1 \|L_\tau u\|_{2,\Omega}^2.$$

Finally, we find

$$\|u\|_{2,2,\Omega} \leq c_4 \|L_\tau u\|_{2,\Omega} \tag{4.2.12}$$

with a constant c_4 that is independent of $u \in H_+^2(\Omega)$ and of $\tau \in [0,1]$. So, the operator L_τ is injective.

The theorem will be proven if we show that the operator L_τ is onto. Then boundedness of the inverse operator follows from the estimate (4.2.12) so that $\|L_\tau^{-1}\| \leq c_4$. First let us show that L_0 is surjective. Indeed, for any $f \in L^2(\Omega)$ there exists $f_m \in \mathcal{M}$ such that $f_m \to f$. Moreover, for each m, there exists $u_m \in H_+^2(\Omega)$ such that $f_m = L_0 u_m$. From estimate (4.2.12) for $\tau = 0$, it follows that $\|u_m - u_k\|_{2,2,\Omega} \leq c_4 \|f_m - f_k\|_{2,\Omega}$ and hence there exists $u \in H_+^2(\Omega)$ such that $u_m \to u$ in $H_+^2(\Omega)$ and by continuity of the operator L_0 we find $L_0 u = f$.

Now, the equation $L_\tau u = f$ can be re-written in the form $(I+A)u = L_0^{-1} f$, where I is the identity operator in $H_+^2(\Omega)$ and $A = \tau L_0^{-1}(L - L_0) : H_+^2(\Omega) \to H_+^2(\Omega)$. By (4.2.11) and (4.2.12), we have $\|A\| \leq \tau c_4 2c_3 = \tau c_5$. We know (von Neumann) that the operator $I + A$ has the bounded inverse operator if $\|A\| < 1$. So, for $\tau \in [0, \tau_1]$ with $\tau_1 = 1/(2c_5)$, $\|A\| \leq 1/2$ and thus for the same τ the operator L_τ is surjective.

We then can represent the operator L_τ as follows: $L_\tau = L_{\tau_1} + (\tau - \tau_1)(L - L_0)$. The main equation takes then the form $u + (\tau - \tau_1) L_{\tau_1}^{-1}(L - L_0)u = L_{\tau_1}^{-1} f$. Repeating the same arguments as in the first step, we can show that L_τ is surjective for all $\tau \in [0, 2\tau_1]$. After a finite number of steps, we will be cover the whole interval $[0,1]$. Theorem 2.7 is proven.

4.2.3 Smoothness of Distributional Solutions

Theorem 2.8. *Let $u \in L^2(\Omega)$ and $f \in L^2(\Omega)$ satisfy the Poisson equation*

$$\triangle u = -f$$

in the sense of distributions, i.e.,

$$\Delta T_u = -T_f \qquad (\Leftrightarrow \int_\Omega u \Delta\varphi dx = -\int_\Omega f\varphi dx, \quad \forall \varphi \in C_0^\infty(\Omega)).$$

Then $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ *and the following estimate is valid:*

$$\|\nabla u\|_{2,\Omega_0} + \|\nabla^2 u\|_{2,\Omega_0} \leq c(\Omega, \Omega_0)(\|f\|_{2,\Omega} + \|u\|_{2,\Omega}) \tag{4.2.13}$$

whenever $\Omega_0 \Subset \Omega$.

Proof. Fix an arbitrary ball $\omega := B(x_0, R) \Subset \Omega$. We can test the identity that appears in the definition of distributional solution with a function φv, where $\varphi \in C^\infty_0(\omega)$ and $v \in \mathcal{E}(\omega)$, and find

$$\int\limits_\omega \varphi u \Delta v dx = -\int\limits_\omega (f'v + g' \cdot \nabla v) dx,$$

where $f' = \varphi f + u\Delta\varphi$ and $g' = 2u \cdot \nabla\varphi$.

We know that there exists a unique function $w \in H^1_0(\omega)$ such that

$$\int\limits_\omega \nabla w \cdot \nabla v dx = -\int\limits_\omega (f'v + g' \cdot \nabla v) dx$$

for any $v \in H^1_0(\omega)$. Since $\mathcal{E}(\omega) \subset H^1_0(\omega)$, we find after integration by parts in the second identity

$$\int\limits_\omega (u\varphi - w)\Delta v dx = 0$$

for any $v \in \mathcal{E}(\omega)$ and therefore for any $v \in H^2_+(\omega)$. By Theorem 2.7, we can find $v \in H^2_+(\omega)$ such that $-\Delta v = u\varphi - w$. This implies that $w = u\varphi$. The function w obeys the estimate

$$\|\nabla w\|_{2,\omega} \leq c(\|f'\|_{2,\omega} + \|g'\|_{2,\omega})$$

From the latter, we can easily deduce the first statement of the theorem for the first derivatives.

Since we know that $\nabla u \in H^1_{loc}(\Omega)$, we can re-write the first identity in the form

$$\int\limits_\omega \varphi u \Delta v dx = -\int\limits_\omega F v dx,$$

where $F = f' - 2\mathrm{div}(u\nabla\varphi) \in L^2(\omega)$. Now, we can find $w \in H^2_+(\omega)$ such that $-\Delta w = F$ or equivalently

$$\int\limits_\omega w \Delta v dx = -\int\limits_\omega F v dx$$

for any $v \in \mathcal{E}(\omega)$. Then we way repeat the above arguments and conclude $w = \varphi u$. By Theorem 2.7, w satisfies the estimate $\|\nabla^2 w\|_{2,\omega} \leq c\|F\|_{2,\omega}$. Selecting a particular function φ and using the previous estimate, we complete the proof of the theorem. □

Theorem 2.9. *Let $b = 0$, $c = 0$, a be a constant matrix and let $u \in L^2(\Omega)$ satisfy the equation $div(a\nabla u) = 0$ in the sense of distributions, i.e.,*

$$\int\limits_{\Omega} u\, div(a\nabla w)dx = 0, \qquad \forall w \in C_0^\infty(\Omega).$$

Then u is infinitely differentiable inside Ω and satisfies the estimate

$$|u(x)| \leq c(n, \nu, \Omega_0, \Omega)\|u\|_{2,\Omega}, \qquad \forall x \in \Omega_0 \Subset \Omega.$$

Proof Step I. Here, we simply repeat arguments of the first step in the proof of the previous statement replacing $-\Delta u$ with $-\mathrm{div} a\nabla u$ on balls and then using covering by balls we can deduce the first energy estimate

$$\int\limits_{\Omega_1} |\nabla u|^2 dx \leq C_1 \int\limits_{\Omega} |u|^2 dx. \tag{4.2.14}$$

Step II Fix an arbitrary Lipschitz subdomain $\Omega_0 \Subset \Omega$ and find a sequence domains Ω_k, $k = 1, 2, ...$, such that $\Omega_0 \Subset ..... \Subset \Omega_{k+1} \Subset \Omega_k \Subset ... \Subset \Omega_1 \Subset \Omega$. According to Step I, $u \in H^1(\Omega_1)$ and estimate (4.2.14) holds. Now, for any $w \in C_0^\infty(\Omega_1)$, we have

$$0 = -\int\limits_{\Omega_1} u\, \mathrm{div}(a\nabla w_{,i})dx = \int\limits_{\Omega_1} u_{,i}\mathrm{div}(a\nabla w)dx, \qquad i = 1, 2, ..., n.$$

We may repeat the same arguments as in Step I, replacing Ω with Ω_1, Ω_1 with Ω_2, and u with $u_{,i}$. This gives the following facts: $u \in H^2(\Omega_2)$ and

$$\int\limits_{\Omega_2} |\nabla^2 u|^2 dx \leq C_2 \int\limits_{\Omega_1} |\nabla u|^2 dx \leq C_1 C_2 \int\limits_{\Omega} |u|^2 dx.$$

So, we can state that $u \in H^k(\Omega_k)$ and

$$\int\limits_{\Omega_0} |\nabla^k u|^2 dx \leq C_k \int\limits_{\Omega} |u|^2 dx$$

for any $k = 1, 2, \ldots$.

Now, let $k = [n/2] + 1$. Then applying the embedding Theorem 1.4 $(k-1)$ times, we get $w \in W^{1,p}(\Omega_0)$ with any $p > n$ if n is even and with $p = \frac{2n}{n-2[n/2]} > n$ if n is odd and the estimate $\|u\|_{p,1,\Omega_0} \leq C\|u\|_{2,\Omega}$ holds. It remains to apply the embedding Theorem 2.5 to get the estimate of the theorem. □

4.2.4 More about Variable Coefficients

In this section, we are going to consider the simplest case of the elliptic equation

$$-\mathrm{div}\, a\nabla u = 0 \tag{4.2.15}$$

in Ω provided that a is a symmetric matrix with bounded measurable entries satisfying ellipticity condition

$$\nu\mathbb{I} \leq a \leq \nu^{-1}\mathbb{I}$$

for some positive ν.

The best known result in this direction is:

Theorem 2.10. *(DeGiorgi-Nash-Moser) Let $b = 0$, $c = 0$ and let $u \in H^1(\Omega)$ satisfy the identity $\mathcal{L}(u, w) = 0$ for any $w \in C_0^\infty(\Omega)$. Then u is Hölder continuous inside Ω with an exponent depending on n and ν only.*

In what follows, we assume about a a bit more:

$$x \mapsto a(x) \tag{4.2.16}$$

is continuous at any point $x \in \Omega$. Since our analysis will be essentially local, we may assume that $\Omega \in \mathbb{R}^n$ is bounded and

$$a \in C(\overline{\Omega}). \tag{4.2.17}$$

Our main result is as follows:

Theorem 2.11. *Let 4.2.17 hold. Let $u \in H^1(\Omega)$ satisfy equation (4.2.15) in the following weak sense*

$$\mathcal{L}(u, v) = \int_\Omega (a\nabla u) \cdot \nabla v dx = 0 \tag{4.2.18}$$

for any $v \in C_0^\infty(\Omega)$. Then for any $0 < \alpha < 1$ and for any $\Omega_0 \Subset \Omega$,

$$u \in C^\alpha(\Omega_0).$$

We start a proof of Theorem 2.11 with auxiliary statements.

Lemma 2.12. *(Morrey) Assume that* $u \in H^1(B(R))$ *and there exist two constants* A *and* $\alpha \in]0,1[$ *such that*

$$\int\limits_{B(x_0,r)} |\nabla u|^2 dx \le A r^{n-2+2\alpha}$$

for any $B(x_0,r) \subset B(R)$. *Then, for any* $0 < \varrho < R$, $u \in C^\alpha(\overline{B}(\varrho))$, *with the estimate*

$$\|u\|_{C^\alpha(\bar{B}(\varrho))} \le C(n,\alpha,\varrho,R)(\sqrt{A} + \|u\|_{1,B(R)}) \tag{4.2.19}$$

PROOF Follows from Lemma 2.2, Hölder inequality, and Poincare-Sobolev inequality, see arguments in the proof of the second embedding theorem.

Lemma 2.13. *Assume that an increasing function* $\Phi : [0,R_0] \to [0,\infty[$ *satisfies the following property*

$$\Phi(r) \le c\Big[\Big(\frac{r}{R}\Big)^n + \varepsilon\Big]\Phi(R)$$

for any $0 < r \le R \le R_0$ *with some positive constants* c *and* ε.

For any $0 < \gamma < n$, *there exists* $\varepsilon_0 = \varepsilon_0(n,\gamma,c)$ *such that if* $\varepsilon \le \varepsilon_0$ *then*

$$\Phi(r) \le c_1(n,\gamma,c)\Big(\frac{r}{R_0}\Big)^{n-\gamma}\Phi(R_0)$$

for all $0 < r < R_0$.

PROOF Take $0 < \tau < 1$ satisfying the condition

$$2c\tau^\gamma \le 1 \tag{4.2.20}$$

and let

$$\varepsilon_0 = \tau^n.$$

Then for $r = \tau^{k+1}R_0$ and $R = \tau^k R_0$, we have

$$\Phi(\tau^{k+1}R_0) \le c(\tau^n + \varepsilon)\Phi(\tau^k R_0) \le$$

$$\le c(\tau^n + \varepsilon_0)\Phi(\tau^k R_0) \le$$

$$\le c\tau^\gamma \tau^{n-\gamma}(1 + \varepsilon_0 \tau^{-n})\Phi(\tau^k R_0) \le$$

$$\leq \tau^{n-\gamma}\Phi(\tau^k R_0).$$

Iterating the last inequality in k starting with $k = 0$, we find

$$\Phi(\tau^k R_0) \leq \tau^{k(n-\gamma)}\Phi(R_0)$$

for any non-negative integer k. Given $0 < r < R_0$, we can find k such that

$$\tau^{k+1}R_0 \leq r < \tau^k R_0$$

which implies

$$\tau^k \leq \frac{r}{\tau R_0}.$$

So,

$$\Phi(r) \leq \Phi(\tau^k R_0) \leq \Big(\frac{r}{\tau R_0}\Big)^{n-\gamma}\Phi(R_0). \qquad \Box$$

PROOF OF THEOREM 2.11 Our proof is based on the so-called method of "frozen" coefficients. Take any ball $B(x_0, R) \Subset \Omega$. Consider the following auxiliary boundary value problem:

$$-\text{div}(a(x_0)\nabla v) = 0 \tag{4.2.21}$$

in $B(x_0, R)$ and

$$v = u \tag{4.2.22}$$

on $\partial B(x_0, R)$. We know that there exists a unique weak solution $v \in H^1(B(x_0, R))$ to boundary value problem (4.2.21) and (4.2.22). Moreover, the solution v is infinitely smooth inside of the ball $B(x_0, R)$ and satisfies the estimate (explain why)

$$\sup_{x\in B(x_0,R/2)} |\nabla v(x)|^2 \leq c(n,\nu)\frac{1}{R^n}\int\limits_{B(x_0,R)} |\nabla v|^2 dx.$$

So, if $0 < r \leq R/2$, then

$$\int\limits_{B(x_0,r)} |\nabla v|^2 dx \leq c\Big(\frac{r}{R}\Big)^n \int\limits_{B(x_0,R)} |\nabla v|^2 dx,$$

if $R/2 < r \leq R$, then

$$\int\limits_{B(x_0,r)} |\nabla v|^2 dx \leq \Big(\frac{2r}{R}\Big)^n \int\limits_{B(x_0,R)} |\nabla v|^2 dx.$$

And finally

$$\int\limits_{B(x_0,r)} |\nabla v|^2 dx \le c\Big(\frac{r}{R}\Big)^n \int\limits_{B(x_0,R)} |\nabla v|^2 dx$$

for all $0 < r \le R$.

Now, we wish to compare our solution u with auxiliary one v:

$$\int\limits_{B(x_0,r)} |\nabla u|^2 dx = \int\limits_{B(x_0,r)} |\nabla(v + u - v)|^2 dx \le$$

$$\le 2 \int\limits_{B(x_0,r)} |\nabla v|^2 dx + 2 \int\limits_{B(x_0,r)} |\nabla(u - v)|^2 dx \le$$

$$\le c\Big(\frac{r}{R}\Big)^n \int\limits_{B(x_0,R)} |\nabla v|^2 dx + 2 \int\limits_{B(x_0,R)} |\nabla(u - v)|^2 dx.$$

Using the same trick $v = (v - u) + u$, we show now

$$\int\limits_{B(x_0,r)} |\nabla u|^2 dx \le c\Big(\frac{r}{R}\Big)^n \int\limits_{B(x_0,R)} |\nabla u|^2 dx + c \int\limits_{B(x_0,R)} |\nabla(u - v)|^2 dx \qquad (4.2.23)$$

for any $0 < r \le R$ with a constant c depending only on n and ν. Now, we need evaluate the second term on the right hand side of the latter inequality. Indeed, according to the definition of weak solutions u and v, we find (explain why)

$$0 = \int\limits_{B(x_0,R)} (a(x)\nabla u - a(x_0)\nabla v) \cdot \nabla(u - v) dx =$$

$$= \int\limits_{B(x_0,R)} \Big((a(x) - a(x_0))\nabla u + a(x_0)\nabla(u - v)\Big) \cdot \nabla(u - v) dx.$$

From the last identity and from the ellipticity condition, one can deduce

$$\nu \int\limits_{B(x_0,R)} |\nabla(u - v)|^2 dx \le \int\limits_{B(x_0,R)} |(a(x) - a(x_0))\nabla u||\nabla(u - v)| dx \le$$

$$\le \Big(\int\limits_{B(x_0,R)} |a(x) - a(x_0)|^2 |\nabla u|^2 dx\Big)^{\frac12} \Big(\int\limits_{B(x_0,R)} |\nabla(u - v)|^2 dx\Big)^{\frac12}$$

and thus

$$\int\limits_{B(x_0,R)} |\nabla(u-v)|^2 dx \leq \frac{1}{\nu^2} \sup_{x\in B(x_0,R)} |a(x)-a(x_0)|^2 \int\limits_{B(x_0,R)} |\nabla u|^2 dx.$$

And (4.2.23) can be transformed into the following one

$$\int\limits_{B(x_0,r)} |\nabla u|^2 dx \leq c(n,\nu)\Big[\Big(\frac{r}{R}\Big)^n +$$

$$+ \sup_{x\in B(x_0,R)} |a(x)-a(x_0)|^2\Big] \int\limits_{B(x_0,R)} |\nabla u|^2 dx \tag{4.2.24}$$

for all $0 < r \leq R$.

We take an arbitrary number $0 < \alpha < 1$ and let $\gamma = 2-2\alpha$. Then, using a constant c from (4.2.24) and the number γ, we can find number ε_0 of Lemma 2.13. This number depends on n, ν, and α only. By uniform continuity, we can find δ such that

$$|a(x)-a(x_0)| \leq \sqrt{\varepsilon_0} \tag{4.2.25}$$

provided $x, x_0 \in \Omega$ and $|x-x_0| \leq \delta$.

Now, let us take a ball $B(z,R_*) \Subset \Omega$ with $R_* < \delta$ and fix it. We then let $R_0 = R_*/4$ and thus, for any $x_0 \in B(z, 3/4R_*)$,

$$B(x_0,R_0) \subset B(z,R_*)$$

and (4.2.25) holds for any $x \in B(x_0,R_0)$. If we let

$$\Phi(r) = \int\limits_{B(x_0,r)} |\nabla u|^2 dx,$$

then we can derive from (4.2.24)

$$\Phi(r) \leq c(n,\nu)\Big[\Big(\frac{r}{R}\Big)^n + \varepsilon_0\Big]\Phi(R)$$

for all $0 < r \leq R \leq R_0$. Now, we are in position to apply Lemma 2.13

$$\Phi(r) \leq c(n,\nu)\Big(\frac{r}{R_0}\Big)^{n-2+2\alpha}\Phi(R_0) \leq$$

$$\leq c(n,\nu)\Big(\frac{r}{R_0}\Big)^{n-2+2\alpha}\int\limits_{\Omega}|\nabla u|^2dx = C_1(n,\nu,\alpha,R_*,\|\nabla u\|_{2,\Omega})r^{n-2+2\alpha}$$

for any $0 < r \leq R_0$.

Now, let us $B(x_0,r)\subset B(z,3/4R_*)$. If $r \leq R_0$, then as it has been shown above $\Phi(x_0,r)\leq C_1 r^{n-2+2\alpha}$. If $r > R_0$, then we argue as follows:

$$\Phi(r)\leq\int\limits_{\Omega}|\nabla u|^2dx\leq\Big(\frac{r}{R_0}\Big)^{n-2+2\alpha}\int\limits_{\Omega}|\nabla u|^2dx =$$

$$= C_2(n,\alpha,R_*,\|\nabla u\|_{2,\Omega})r^{n-2+2\alpha}.$$

So, for $A=\max\{C_1,C_2\}$,

$$\int\limits_{B(x_0,r)}|\nabla u|^2dx\leq Ar^{n-2+2\alpha}$$

provided $B(x_0,r)\subset B(z,3/4R_*)$. By Lemma 2.12,

$$\|u\|_{C^\alpha(\overline{B}(z,R_*/2))}\leq C(n,\nu,\alpha,R_*/2,3/4R_*)(\sqrt{A}+\|u\|_{1,\Omega}) =$$

$$= c_0(n,\nu,\alpha,R_*,\|u\|_{2,1,\Omega}).$$

Given $\Omega_0\Subset\Omega$, let us take $r=\frac{1}{3}\min\{\delta,\operatorname{dist}(\Omega_0,\partial\Omega)\}$. Then

$$\|u\|_{C^\alpha(\overline{B}(z,r/2))}\leq C := c_0(n,\nu,\alpha,r,\|u\|_{2,1,\Omega})$$

for any $z\in\Omega_0$. Obviously $\|u\|_{\infty,\Omega_0}\leq C$. Next, let $x_0,y_0\in\Omega_0$. If $|x_0-y_0| < r/2$, then $|u(x_0)-u(y_0)|\leq C|x_0-y_0|^\alpha$. If $|x_0-y_0|\geq r/2$, then

$$|u(x_0)-u(y_0)|\leq 2\|u\|_{\infty,\Omega_0}\leq 2C\Big(\frac{|x_0-y_0|}{r/2}\Big)^\alpha$$

and thus

$$\|u\|_{C^\alpha(\overline{\Omega}_0)}\leq(1+2^{1+\alpha}r^{-\alpha})C.$$

□

In fact, if we assume that

$$a\in C^\alpha(\overline{\Omega}) \tag{4.2.26}$$

for some $0<\alpha<1$, then Theorem 2.11 can be essentially improved.

Theorem 2.14. *Let (4.2.26) and (4.2.18) hold. Then*

$$\nabla u \in C^{\beta}(\overline{\Omega}_0)$$

for any $0 < \beta < \alpha$ *and for any* $\Omega_0 \Subset \Omega$.

PROOF The plan of the proof is the same as in the previous theorem. We "freeze" coefficients and consider auxiliary problem (4.2.21). But, in this case, the different estimate for solutions to elliptic equations with constant coefficients is used. Namely,

$$\int\limits_{B(x_0,r)} |\nabla v - [\nabla v]_{x_0,r}|^2 dx \leq c(n,\nu)\Big(\frac{r}{R}\Big)^{n+2} \int\limits_{B(x_0,R)} |\nabla v - [\nabla v]_{x_0,R}|^2 dx$$

for any $0 < r \leq R$. Here, we use abbreviation $[f]_{x_0,R} := [f]_{B(x_0,R)}$. This estimate can be deduced from Theorem 2.9 (exercise). We can then repeat the same arguments as in Theorem 2.11 and get

$$\int\limits_{B(x_0,r)} |\nabla u - [\nabla u]_{x_0,r}|^2 dx \leq c\Big(\frac{r}{R}\Big)^{n+2} \int\limits_{B(x_0,R)} |\nabla u - [\nabla u]_{x_0,R}|^2 dx +$$

$$+c \int\limits_{B(x_0,R)} |\nabla (u-v)|^2 dx.$$

For the error $v - u$, we have the same estimate

$$\int\limits_{B(x_0,R)} |\nabla (u-v)|^2 dx \leq \frac{1}{\nu^2} \sup_{x \in B(x_0,R)} |a(x) - a(x_0)|^2 \int\limits_{B(x_0,R)} |\nabla u|^2 dx.$$

But, since a is Hölder continuous,

$$\int\limits_{B(x_0,R)} |\nabla (u-v)|^2 dx \leq cR^{2\alpha} \int\limits_{B(x_0,R)} |\nabla u|^2 dx = cR^{2\alpha}\Phi(x_0,R) \qquad (4.2.27)$$

and

$$\int\limits_{B(x_0,R)} |\nabla (u-v)|^2 dx \leq cR^{2\alpha}\Psi(x_0,R) + cR^{2\alpha}|B(R)|(|[\nabla u]_{x_0,R})|^2 \leq \qquad (4.2.28)$$

$$\leq cR^{2\alpha}\Psi(x_0,R) + cR^{2\alpha}\Phi(x_0,R),$$

where

$$\Psi(x_0,R) := \int\limits_{B(x_0,R)} |\nabla u - [\nabla u]_{x_0,R}|^2 dx.$$

Then repeating arguments of the previous theorem and using arguments (4.2.27) and (4.2.28), we find

$$\Phi(x_0,r) \leq c\Big[\Big(\frac{r}{R}\Big)^n + R^{2\alpha}\Big]\Phi(x_0,R) \tag{4.2.29}$$

and

$$\Psi(x_0,r) \leq c\Big[\Big(\frac{r}{R}\Big)^{n+2} + R^{2\alpha}\Big]\Psi(x_0,R) + cR^{2\alpha}\Phi(x_0,R) \tag{4.2.30}$$

for any $0 < r \leq R$.

Now, fix an arbitrary subdomain $\Omega_0 \Subset \Omega$ and let

$$R_0 := \frac{1}{2}\min\{\varepsilon_0^{\frac{1}{2\alpha}}, \mathrm{dist}(\Omega_0, \partial\Omega\}.$$

Then, by Lemma 2.13, we can deduce from (4.2.29)

$$\Phi(x_0,R) \leq cR^{n-2(\alpha-\beta)}$$

for any $0 < R \leq R_0$. But then (4.2.28) gives us:

$$\Psi(x_0,r) \leq c\Big[\Big(\frac{r}{R}\Big)^{n+2} + R^{2\alpha}\Big]\Psi(x_0,R) + cR^{n+2\beta} \tag{4.2.31}$$

for any $0 < r \leq R \leq R_0$. Now, we need a generalisation of Lemma 2.13:

Lemma 2.15. *Let $\Xi : [0,R_0] \to [0,\infty[$ be an increasing function having the following property:*

$$\Xi(r) \leq c\Big[\Big(\frac{r}{R}\Big)^\alpha + \varepsilon\Big]\Xi(R) + AR^\beta$$

for any $0 < r \leq R \leq R_0$ with some positive constants c, A, α, β, and ε satisfying the condition $\alpha > \beta$. Show that there exists a constant $\varepsilon_1(c,A,\alpha,\beta)$ such that if $\varepsilon \leq \varepsilon_1$, then

$$\Xi(r) \leq c_1\Big[\Big(\frac{r}{R_0}\Big)^\beta \Xi(R_0) + Ar^\beta\Big]$$

for all $0 < r \leq R_0$ and for some positive constant $c_1(c,\alpha,\beta)$.

So, there exists $\varepsilon_1 > 0$ such that

$$\Psi(x_0, R) \leq A_1 R^{2\beta}$$

for any $0 < R \leq R_1 := \frac{1}{2}\min\{\varepsilon_1^{\frac{1}{2\alpha}}, R_0\}$, for any $x_0 \in \Omega_0$, and for some positive constant A_1. Now, the statement of the theorem follows from Campanato's condition, see Lemma 4.2.4. $\square$

Appendix A

Functional Analysis Background

A.1 Normed Spaces

Definition 1.1. *Let X be a real or complex vector space. A* <u>*norm*</u> *on X is a function $\| \cdot \| : X \to \mathbb{R}$ satisfying*

$$\begin{array}{lll} (N1) & \|x\| \geq 0, \quad \|x\| = 0 \Leftrightarrow x = 0 & \\ (N2) & \|\alpha x\| = |\alpha| \|x\| & \forall x \in X, \quad \forall \alpha \in \mathbb{R} (or \mathbb{C}) \\ (N3) & \|x + y\| \leq \|x\| + \|y\| & \forall x, y \in X. \end{array}$$

$(X, \| \cdot \|)$ is then called a *normed* space.

Remark 1.2. *If instead of $(N1)$ one only has*

$$(SN1) \quad \|x\| \geq 0 \quad \text{and} \quad x = 0 \Rightarrow \|x\| = 0$$

then $\|x\|$ is a called <u>*semi-norm*</u>.

Examples

I. $\mathbb{R}^N$, $\quad x = (x_1, x_2, ..., x_N) = (x_i) \in \mathbb{R}^N$

$$\begin{aligned} \|x\|_1 &= \sum_{n=1}^{N} |x_n| \\ \|x\|_\infty &= \sup_{1 \leq n \leq N} |x_n| \\ \|x\|_2 &= \Big(\sum_{n=1}^{N} |x_n|^2 \Big)^{\frac{1}{2}}. \end{aligned}$$

II. l^p, $\quad x = (x_1, x_2, ...) = (x_i) \in l^p$, $1 \leq p \leq \infty$,

$$\begin{aligned} \|x\|_p &= \Big(\sum_{n=1}^{\infty} |x_n|^p\Big)^{\frac{1}{p}}, \quad 1 \leq p < \infty \\ \|x\|_\infty &= \sup_{1 \leq n < \infty} |x_n|, \quad p = \infty. \end{aligned}$$

III. $f \in C(\overline{\Omega})$, $f : \overline{\Omega} \to \mathbb{R}$,

$$\begin{aligned} \|f\|_\infty &= \sup_{x \in \overline{\Omega}} |f(x)| \\ \|f\|_2 &= \Big(\int_\Omega |f(x)|^2 dx\Big)^{\frac{1}{2}}. \end{aligned}$$

IV. $C^1([a,b])$, $f : [a,b] \to \mathbb{R}$,

$$\begin{aligned} \|f\|_{C^1([a,b])} &= \sup_{x \in [a,b]} (|f(x)| + |f'(x)|) \\ \|f\| &= \sup_{x \in [a,b]} |f'(x)| \text{ (semi-norm)}. \end{aligned}$$

Definition 1.3. *Let X be a real or complex vector space. An inner (scalar) product on X is a function $(\cdot, \cdot) : X \times X \to \mathbb{R}$ (or C) satisfying*

$$\begin{array}{lll} (I1) & (x,x) \geq 0, & (x,x) = 0 \Leftrightarrow x = 0 \\ (I2) & (x,y) = \overline{(y,x)} & \forall x, y \in X \\ (I3) & x \mapsto (x,y) & \textit{is linear for each fixed } y \in X. \end{array}$$

$(X, (\cdot, \cdot))$ is then called a *pre-Hilbert* space.

Remark 1.4. *If instead of $(I1)$ one only has*

$$(SI1) \qquad (x,x) \geq 0 \tag{A.1.1}$$

then $(\cdot, \cdot)$ is a called semi-indefinite scalar product.

From the Cauchy-Schwarz inequality

$$|(x,y)| \leq (x,x)^{\frac{1}{2}} (y,y)^{\frac{1}{2}},$$

it follows

Lemma 1.5. *If $(\cdot,\cdot)$ is (semi-indefinite) scalar product, then $\|x\| = (x,x)^{\frac{1}{2}}$, $x \in X$, is (semi-) norm.*

EXAMPLES
I. $(x,y) = \sum\limits_{i=1}^{\infty} x_i y_i$ is a scalar product on l^2.
II. $(f,g) = \int\limits_{\Omega} f(x)g(x)dx$, $\forall f, g \in C(\overline{\Omega})$, is a scalar product on $C(\overline{\Omega})$.

A.2 Completeness

Definition 2.1. *A Banach space is a normed space $(X, \|\cdot\|)$ that is complete: if $\{x^{(i)}\}$ is a Cauchy sequence, i.e., $\|x^{(i)} - x^{(j)}\| \to 0$ as i and j tend to ∞, then $x^{(i)}$ is convergent, i.e., there exists $x \in X$ such that $\|x^{(i)} - x\| \to 0$ as i tends to ∞.*

Definition 2.2. *A pre-Hilbert space is a Hilbert space if it is a Banach space with respect to the norm $\|x\| = (x,x)^{\frac{1}{2}}$.*

EXAMPLES
I. The space of all continuous functions $f : \overline{\Omega} \to \mathbb{R}$ equipped with $\|\cdot\|_\infty$ is denoted by $C(\overline{\Omega})$. It is a Banach space. The proof relies upon two facts: convergence with respect to the norm $\|\cdot\|_\infty$ is equivalent to uniform convergence and on the Weierstrass theorem on the limit of uniformly converging continuous functions.
II. Now assume that the space of continuous functions is equipped with $\|\cdot\|_2$. This is an example of a normed space that is not Banach one. To see that let $n = 1$, $\Omega = (-1,1)$, and $f_i(x) = ix$ if $x \in [-1/i, 1/i]$, $f_i(x) = -1$ if $x \in [-1, -1/i)$, and $f_i(x) = 1$ if $x \in (1/i, 1]$. It is a Cauchy sequence (explain why). Suppose that there exists $f \in C(\overline{\Omega})$ such that

$$\int\limits_{-1}^{1} |f_i(x) - f(x)|^2 dx \to 0 \qquad \text{as} \quad i \to \infty.$$

Since f_i and f are uniformly bounded, it is easy to see (explain why) that $f(x) = -1$ for $x \in (-1,0)$ and $f(x) = 1$ for $x \in (0,1)$, which means that f is discontinuous at $x = 0$.

A.3 Separability

In the normed space $(X, \|\cdot\|)$, topology is defined with the help of the metric generated by the norm: $\varrho(x, y) = \|x - y\|$ is the distance between $x \in X$ and $y \in X$. With this metric, we can define open ball and neighborhoods, open and closed sets, interior and closure of a given sets, etc.

Definition 3.1. *Let $(X, \|\cdot\|)$ be a normed space. It is separable, if there exists a countable set $A \in X$ with the following property. For any $x \in X$ and for any positive number ε, there exists $a \in A$ such that $\|x - a\| < \varepsilon$.*

EXAMPLES
I. l^p, with $1 \leq p < \infty$, is separable but l^∞ is not.
II. $C(\overline{\Omega})$ is separable. The idea of a proof in the simplest case $\Omega = (0, 1)$ is as follows. Consider a countable set A consisting of all piece-wise linear functions on $[0, 1]$ with a finite number of vertices at points having rational coordinates in the plane and show that it is required.

A.4 Compactness

Definition 4.1. *Let $(X, \|\cdot\|)$ be a normed space. We say that $K \subset X$ is a compact set of X if any open covering of K contains a finite subcovering. We say that the set K is precompact if its closure is compact.*

Definition 4.2. *Let $(X, \|\cdot\|)$ be a normed space. We say that $K \subset X$ is a sequentially compact set of X if any sequence of K contains a converging subsequence whose limit belongs to K.*

Theorem 4.3. *Let $(X, \|\cdot\|)$ be a normed space and $K \subset X$. K is compact if and only if K is sequentially compact.*

Lemma 4.4. *Let $(X, \|\cdot\|)$ be a normed space and $K \subset X$ be compact. Then K is bounded and closed.*

Theorem 4.5. *(Hausdorff) Let $(X, \|\cdot\|)$ be a Banach space. K is precompact if and only if for $\varepsilon > 0$ there exists a finite ε-net, i.e., $K \in \bigcup_{j=1}^{m} B_X(x_j, \varepsilon)$ for some $x_j \in X$. Here, $B_X(x, \varrho)$ is an open ball of X with radius ϱ centered at point x.*

EXAMPLES
I. Finite-dimensional spaces

Lemma 4.6. *(Heine-Borel) Any bounded closed set of a finite-dimensional space is compact.*

II. CLAIM: *The unit closed ball of l^2 is not compact.* Indeed, let $x^{(j)} = (x_i^{(j)})$ with $x_i^{(j)} = 0$ if $i \neq j$ and $x_i^{(j)} = 1$ if $i = j$. Since $\|x^{(j)} - x^{(i)}\|_2 = \sqrt{2}$ if $i \neq j$, the sequence $x^{(j)}$ does not contain a converging subsequence.
III. $C(\overline{\Omega})$,

Theorem 4.7. *(Ascoli-Arzela) A sequence $\{f^{(j)}\}_{j=1}^{\infty}$ of $C(\overline{\Omega})$ contains a converging subsequence if and only if $\{f^{(j)}\}_{j=1}^{\infty}$ has the following properties:*
(i) $\{f^{(j)}\}_{j=1}^{\infty}$ is uniformly bounded, i.e., $\sup_j \|f^{(j)}\|_\infty < \infty$
(ii) $\{f^{(j)}\}_{j=1}^{\infty}$ is equi-continuous, i.e., for any $\varepsilon > 0$, there exists $\tau > 0$ such that $|f^{(j)}(x) - f^{(j)}(y)| < \varepsilon$ for any natural j and for any $x, y \in \overline{\Omega}$ with $|x - y| < \tau$.

A.5 Linear Operators

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be a normed space. A linear operator $A : X \to Y$ is bounded if

$$\|Ax\|_Y \leq C\|x\|_X, \qquad \forall x \in X.$$

Lemma 5.1. *A linear operator $A : X \to Y$ is continuous on X if and only if it is bounded.*

The least constant for which the latter inequality is called the norm of A and denoted as follows

$$\|A\| := \inf\{C : \ \|Ax\|_Y \leq C\|x\|_X, \quad \forall x \in X\}.$$

Moreover,

$$\|A\| = \sup\{\|Ax\|_Y : \ \|x\|_X \leq 1\}.$$

Theorem 5.2. *(Uniform Boundedness Principle, Banach-Steinhaus)) Let $A_n : X \to Y$ be a sequence of linear operators and x be a B-space. Then*

$$\sup_n \|A_n\| < \infty$$

if and only if

$$\sup_n \|A_n x\| < \infty$$

for each $x \in X$.

The linear space $B(X,Y)$ of all linear bounded operators : $X \to Y$ is a B-space itself provided Y is B-space with respect to the operator norm defined above.

A.6 Duality

Let $(X, \|\cdot\|_X)$ be a normed space. A linear operator $T : X \to \mathbb{R}$ (or $\mathbb{C}$) is called a linear functional and is bounded if

$$|T(x)| \leq C\|x\|_X, \qquad \forall x \in X.$$

Lemma 6.1. *Linear functional $T : X \to \mathbb{R}$ (or $\mathbb{C}$) is continuous on X if and only if it is bounded.*

Definition 6.2. *A dual space X^* is the space of all continuous linear functional on X. We denote by x^* its elements and the action x^* on x is denoted by $x^*(x)$ or $< x^*, x >$.*

X^* is a Banach space with respect to the dual norm

$$\|x^*\|_{X^*} = \sup\{x^*(x) : \ \|x\|_X = 1\}.$$

Examples: representation of linear functionals in certain spaces
I. $\mathbb{R}^N$, there exists *isometric* (preserves norm) *isomorphism* (linear mapping: one-to-one and onto)$\pi : (\mathbb{R}^N)^* \to \mathbb{R}^N$ so that $\pi x^* = x$ (isometry: $\|x\|_{\mathbb{R}^N} = \|x^*\|_{(\mathbb{R}^N)^*}$) with

$$x^*(y) = \sum_{i=1}^{N} x_i y_i \qquad \forall y \in \mathbb{R}^N.$$

So, we have $(\mathbb{R}^N)^* \cong \mathbb{R}^N$ (up to isometric isomorphism). In what follows, in such cases, we are going to use a simpler notation $(\mathbb{R}^N)^* = \mathbb{R}^N$.
II. Let U be a Hilbert space with a scalar product $(\cdot,\cdot)$, there exists isometric isomorphism $U^* \to U$, so that $\pi u^* = u$ with

$$u^*(v) = (u, v) \qquad \forall v \in U.$$

So, we have $U^* = U$.

III. l^p with $1 \le p < \infty$, there exists isometric isomorphism $\pi : (l^p)^* \to l^{p'}$, $p' = \frac{p}{p-1}$, so that $\pi x^* = x$ with

$$x^*(y) = \sum_{i=1}^{\infty} x_i y_i \qquad \forall y \in l^p.$$

So, we have $(l^p)^* = l^{p'}$.

By definition $X^{**} = (X^*)^*$, each $x \in X$ generates a functional x_x^{**} in the following way

$$x_x^{**}(x^*) = x^*(x), \qquad \forall x^* \in X^*.$$

So, the latter identity defines a mapping $\tau : X \to X^{**}$ so that $\tau x = x_x^{**}$. It is known that τ is isometric isomorphism from X onto $\widetilde{X} = \tau(X)$. Obviously, $\widetilde{X}$ is a subspace of X^{**}. A space X is called *reflexive* if $X^{**} = \widetilde{X}$.

EXAMPLES. $\mathbb{R}^N$, U, and l^p with $1 < p < \infty$ are reflexive but l^1, l^∞, and $C(\overline{\Omega})$ are not.

Let $(X, \|\cdot\|_X)$ be a B-space (Banach space). Let $x^{(j)}$ be a sequence in X. We say that $x^{(j)}$ converges to $x \in X$ *strongly* as $j \to \infty$ ($x^{(j)} \to x$) if $\|x^{(j)} - x\|_X \to 0$. We say that $x^{(j)}$ converges to $x \in X$ *weakly* ($x^{(j)} \rightharpoonup x$) if $x^*(x^{(j)}) \to x^*(x)$ as $j \to \infty$ for any $x^* \in X^*$. Finally, we say that a sequence $x^{*(j)} \in X^*$ converges to $x^* \in X^*$ *weakly*-$(*)$ as $j \to \infty$ ($x^{*(j)} \overset{*}{\rightharpoonup} x^*$) if $x^{*(j)}(x) \to x^*(x)$ for any $x \in X$.

Remark 6.3. *A consequence of the Banach-Steinhaus theorem, see 5.2, is as follows. Assume that $x^{(j)} \rightharpoonup x$ ($x^{*(j)} \overset{*}{\rightharpoonup} x^*$), then*

$$\sup_j \|x^{(j)}\|_X (\sup_j \|x^{*(j)}\|_{X^*}) < \infty.$$

Moreover,

$$\liminf_{j\to\infty} \|x^{(j)}\|_X (\|x^{*(j)}\|_{X^*}) \ge \|x\|_X (\|x^*\|_{X^*}).$$

EXAMPLES

I. l^p with $1 \le p < \infty$.

$$x^{(j)} \rightharpoonup x \Leftrightarrow \sum_{i=1}^{\infty} y_i x_i^{(j)} \to \sum_{i=1}^{\infty} y_i x_i \quad \forall y = (y_i) \in l^{p'}$$

II. l^∞.

$$x^{(j)} \overset{*}{\rightharpoonup} x \Leftrightarrow \sum_{i=1}^{\infty} y_i x_i^{(j)} \to \sum_{i=1}^{\infty} y_i x_i \quad \forall y = (y_i) \in l^1$$

Strong convergence implies weak convergence but opposite in general is wrong. Indeed, consider l^2 with the sequence described by the claim right after Lemma 4.6. In fact, this sequence converges weakly to zero (explain why). The weak convergence implies the strong convergence in finite-dimensional spaces. Sequences of X^*, converging weakly, converges weakly-($*$), the opposite statement is true, in general, in reflexive B-spaces only.

We know that in finite-dimensional spaces bounded sequences are precompact, i.e., any bounded sequence contains a convergent subsequence. For the infinite-dimensional case, such a statement in general is not true. However, if we replace strong convergence by weak or weak-($*$) convergence, the corresponding statement turns out to be true in a number of cases interesting for applications.

Theorem 6.4. *(Banach-Alaoglu) Let $(X, \|\cdot\|_X)$ be a separable B-space. Suppose that*

$$\sup_j \|x^{*(j)}\|_{X^*} < \infty.$$

Then there exists a subsequence $x^{*(j_k)}$ such that

$$x^{*(j_k)} \overset{*}{\rightharpoonup} x^* \in X^*$$

as $k \to \infty$.

PROOF On Sheet 1.

We can get rid of separability, if we assume that X is reflexive.

Theorem 6.5. *Let $(X, \|\cdot\|_X)$ be a reflexive B-space. Then any bounded sequence in X contains a weakly converging subsequence.*

Under the assumption that X^* is separable, the latter statement easily follows from Theorem 6.4 applied to $X^{**} = X$.

A.7 Fredholm Alternative

Definition 7.1. *Let U be a Hilbert space. An operator $K : U \to U$ is compact or completely continuous if the image (under the action of K) of any bounded set is precompact.*

Theorem 7.2. *Let $K : U \to U$ be compact. Either the non-homogeneous equation*

$$u - Ku = f$$

is uniquely solvable for any $f \in U$ or else the equation

$$u - Ku = 0$$

has non-trivial (non-zero) solutions.

PROOF Suppose that homogeneous equation has the only trivial solution and let us show that the non-homogeneous equation has a solution for any $f \in U$, i.e., $V := (I - K)(U) = \{v \in U : \; v = u - Ku, \, u \in U\} = U$.

Let us first show that V is closed. Let $v_m \in V$ and $v_m \to v$. We shall prove that $v \in V$. By definition, there exists $u_m \in U$ such that $v_m = u_m - Ku_m$. Let us show that u_m is bounded. If not, WLOG, we may assume that $\|u_m\| \to \infty$. Setting $v'_m = v_m/\|u_m\|$ and $u'_m = u_m/\|u_m\|$, we observe that $v'_m \to 0$ and u'_m is bounded. If so, by Theorem 6.5, there exists s subsequence u'_{m_k} such that $u'_{m_k} \rightharpoonup u' \in U$. Passing to the limit in the identity $v'_{m_k} = u'_{m_k} - Ku'_{m_k}$, we find $u' - Ku' = 0$ and thus by assumption $u' = 0$. On the other hand, since K is compact, $Ku'_{m_k} \to Ku'$ and, therefore, $u'_{m_k} \to u'$, which implies (since $\|u'_m\| = 1$) $\|u'\| = 1$. This is a contradiction. Since u_m is bounded, we can apply Theorem 6.5 again and get $u_{m_k} \to u \in U$. Then taking the limit in $v_{m_k} = u_{m_k} - Ku_{m_k}$ and find that $v \in V$.

Assume $V \subset U$ (strong inclusion). Now, let us construct a sequence of subspaces of U, letting $V_k = (I - K)(V_{k-1})$, $k = 1, 2, ...$, and $V_0 = V$. Since operator $I - K$ is one-to-one, we have strong inclusion $V_k \subset V_{k-1}$, $k = 1, 2, ...$ Indeed, to this end, it is sufficient to show that $V_2 = (I - K)^2(V_0) \subset V_1 = (I - K)(V_0)$. Assume that $V_2 = V_1$. We know that there exists $u \in V_0 \backslash V_1$. But since $(I - K)u \in V_1 = V_2$, there exists $u_1 \in U$ such that $(I - K)u = (I - K)^2 u_1$. Since the operator $I - K$ is one-to-one, we find $u = (I - K)u_1 \in V_1$. This is a contradiction.

Next, decompose V_k into an orthogonal sum so that $V_k = V_{k+1} \oplus V_{k+1}^{\perp}$, $V_{k+1}^{\perp} := \{w \in V_k : \; (w, v) = 0 \; \forall v \in V_{k+1}\}$. We can then select a sequence $w_k \in V_{k+1}^{\perp}$ such that $\|w_k\| = 1$. Observing, for $k > l$, $w_k - Kw_k \in V_{l+1}$, $w_l - Kw_l \in V_{l+1}$, and $w_k \in V_{l+1}$, we find

$$Kw_l - Kw_k = w_l + \alpha,$$

where $\alpha = -w_k + (w_k - Kw_k) - (w_l - Kw_l) \in V_{l+1}$ and thus

$$\|Kw_l - Kw_k\|^2 = \|w_l\|^2 + \|\alpha\|^2 \geq 1.$$

Since sequence w_k is bounded, sequence Kw_k must be precompact, which contradicts the last estimate. □

What happens if the equation $u = Ku$ has non-trivial solutions? First, as it has been shown in the proof of the Fredholm Alternative, the set $(I-K)(U)$ is closed and thus from the general operator theory it follows that

$$(I-K)(U) = (\ker(I-K^*))^\perp,$$

where the adjoint operator K^* is defined so that

$$(Ku, v) = (u, K^*v) \qquad \forall u, v \in U,$$

I stands for the identity operator in U, and

$$(\ker(I-K^*))^\perp = \{v \in U : \ (u, v) = 0 \ \forall u \in U \ u = K^*u\}.$$

In other words, our non-homogeneous equation $u - Ku = f$ has a solution if and only if $(f, v) = 0$ for any $v \in U$ satisfying the homogeneous equation $v = K^*v$.

Appendix B

Lebesgue's Integration

B.1 Lebesgue's Measure

Let $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ with $a_i \leq b_i$, $i = 1, 2, ..., n$. $\overset{\circ}{P}(a, b) = \prod_{i=1}^{n}]a_i, b_i[$ is an *open parallelepiped*, $\overline{P}(a, b) = \prod_{i=1}^{n} [a_i, b_i]$ is a *closed parallelepiped*. Any set $P(a, b)$ satisfying

$$\overset{\circ}{P}(a, b) \subseteq P(a, b) \subseteq \overline{P}(a, b)$$

is a *parallelepiped* (n-dimensional).

The volume of $P(a, b)$ is $\mathrm{vol}(P(a, b)) := \prod_{i=1}^{n} (b_i - a_i)$.

Definition 1.1. *An (n-dimensional) outer measure of a set $E \in \mathbb{R}^n$ is*

$$\mu^*(E) := \inf\{\sum_{i=1}^{\infty} vol(\overset{\circ}{P}_i) : \ E \subseteq \bigcup_{i=1}^{\infty} \overset{\circ}{P}_i\}$$

Lemma 1.2.

$$\begin{aligned} (i) &\quad \mu^*(E) \in [0, \infty], \qquad E \subseteq \mathbb{R}^n, \\ (ii) &\quad \mu^*(\emptyset) = 0, \\ (iii) &\quad \mu^* \ is \ \sigma - subadditive: \end{aligned}$$

$$\mu^*(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu^*(E_i).$$

Definition 1.3. *A set $E \in \mathbb{R}^n$ is measurable (Lebesgue measurable) if for any $\varepsilon > 0$ there exists an open set $\mathcal{O}_\varepsilon$ such that $E \subseteq \mathcal{O}_\varepsilon$ and $\mu^*(\mathcal{O}_\varepsilon \setminus E) < \varepsilon$. We then call $\mu^*(E)$ the Lebesgue measure of E and denote it by $\mu(E)$.*

The Lebesgue measure is σ-additive, i.e., if E_i, $i = 1, 2, ...$, are measurable and disjoint ($E_i \cap E_j = \emptyset$, $i \neq j$) then

$$\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i).$$

The family of measurable sets in $\mathbb{R}^n$ is a σ-algebra that contains all open sets and all null sets (E is a null set if for any positive ε there exists a countable covering of E by open parallelepipeds whose summary volume is less than ε).

B.2 Measurable Functions

Let E be a measurable set in $\mathbb{R}^n$. A function $f : E \to \overline{\mathbb{R}}$ is measurable if for $\alpha \in \overline{\mathbb{R}}$ the set $\{x \in E : \ f(x) \geq \alpha\}$ is measurable.

Definition 2.1. *(i) $f_m \to f$ almost everywhere (a.e.) in E as $m \to \infty$ if $f_m(x) \to f(x)$ for almost all (a.a.) $x \in E$.*
(ii) Let $\{f_m\}_{m=1}^{\infty}$ and f be measurable and a.e. finite in E. $f_m \to f$ in measure if for any $\varepsilon > 0$

$$\lim_{m\to\infty} \mu\{x \in E : \ |f_m(x) - f(x)| \geq \varepsilon\} = 0.$$

Lemma 2.2. *Let f_m is a sequence of measurable functions in E. If f_m converges to f a.e., then f is measurable in E.*

Theorem 2.3. *(Lebesgue) Let $\{f_m\}_{m=1}^{\infty}$ and f be measurable and a.e. finite in E. Assume that $\mu(E)$ is finite and f_m converges to f a.e. in E. Then f_m converges to f in measure.*

Theorem 2.4. *(Riesz) Let $\{f_m\}_{m=1}^{\infty}$ and f be measurable and a.e. finite in E. Assume that $\mu(E)$ is finite and f_m converges to f in measure in E. Then there exists a subsequence f_{m_k} converging to f a.e.*

B.3 Lebesgue's Integral

Let E be measurable with $\mu(E) < \infty$. A *partition* T is a finite family of disjoint measurable sets whose union is E, i.e., $T = \{E_j\}_{j=1}^m$, $E_i \cap E_j = \emptyset$ if $i \neq j$, and $E = \bigcup_{j=1}^m E_j$.

For a bounded function $f : E \to \mathbb{R}$, we let

$$S(T) = \sum_{k=1}^m M_k \mu(E_k), \quad s(T) = \sum_{k=1}^m m_k \mu(E_k),$$

$$M_k = \sup_{x \in E_k} f(x), \quad m_k = \inf_{x \in E_k} f(x).$$

Then we define *upper* and *lower* Lebesgue integrals:

$$\overline{I}(E) = \inf_T S(T) \qquad \underline{I}(E) = \sup_T s(T).$$

Definition 3.1. *A bounded $f : E \to \mathbb{R}$ is called Lebesgue integrable or integrable over set E if upper and lower integrals coincide. The corresponding value is called Lebesgue integral of f over E and denoted by*

$$\int_E f(x) d\mu(x) \quad (\textit{or simply} \int_E f(x) dx).$$

Theorem 3.2. *Any bounded measurable function is integrable over bounded measurable set.*

Next step is to define Lebesgue's integral for non-negative measurable functions. To this end, let us introduce a truncation of $f \geq 0$ as $f_N(x) = \min\{N, f(x)\}$. By Theorem 3.2, f_N is integrable and a sequence

$$I_N = \int_E f_N(x) dx$$

is increasing as $N \to \infty$.

Definition 3.3. *Let f be non-negative and measurable in E. If I_N is a bounded sequence, then f is integrable in E. $\lim_{N \to \infty} I_N$ is called Lebesgue's integral of f in E and denoted by the same symbols as in Definition 3.1.*

In the same way, we can define Lebesgue's integral of non-negative measurable function $f : E \to \overline{\mathbb{R}}$ for any measurable set E (not necessarily having bounded measure). Let $R > 0$ and define $f_R(x) = f(x)$ if $x \in E \cap B(R)$ and $f_R(x) = 0$ if $x \in E \setminus B(R)$. Then, we can consider

$$I_R = \int\limits_{E \cap B(R)} f(x)dx.$$

Definition 3.4. *Let f be non-negative and measurable in E. If I_R is a bounded sequence, then f is integrable in E. $\lim\limits_{R\to\infty} I_R$ is called Lebesgue's integral of f in E and denoted by the same symbols as in Definition 3.1.*

For arbitrary measurable function $f : E \to \overline{\mathbb{R}}$, defined in arbitrary measurable set E, we proceed as follows. Setting

$$f_+ = \frac{1}{2}(|f| + f), \qquad f_- = \frac{1}{2}(|f| - f).$$

Definition 3.5. *A measurable function $f : E \to \overline{\mathbb{R}}$ is integrable in E if f_+ and f_- are integrable there and*

$$\int\limits_E f(x)dx = \int\limits_E f_+(x)dx - \int\limits_E f_-(x)dx$$

is Lebesgue's integral of f in E.

Theorem 3.6. *f is integrable in E if and only if $|f|$ is integrable in E and*

$$\Big|\int\limits_E f(x)dx\Big| \leq \int\limits_E |f(x)|dx.$$

We denote by $\mathcal{L}^1(E)$, the linear space of all functions integrable in E. Let us list some properties of Lebesgue's integral.

Theorem 3.7. *(Absolute continuity of Lebesgue's integral as a function of sets) Let $f \in \mathcal{L}^1(E)$. For any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$\Big|\int\limits_{E_0} f(x)dx\Big| \leq \int\limits_{E_0} |f(x)|dx < \varepsilon$$

provided $\mu(E_0) < \delta(\varepsilon)$ and $E_0 \subseteq E$.

Theorem 3.8. *(σ-additivity) Let $E = \bigcup_{k=1}^{\infty} E_k$, $E_k \cap E_m = \emptyset$ if $k \neq m$.*
(i) If $f \in \mathcal{L}^1(E)$, then $f \in \mathcal{L}^1(E_k)$ for any k and

$$\int_E f(x)dx = \sum_{k=1}^{\infty} \int_{E_k} f(x)dx. \tag{B.3.1}$$

(ii) If $f \in \mathcal{L}^1(E_k)$ for any k and

$$\sum_{k=1}^{\infty} \int_{E_k} |f(x)|dx < \infty,$$

then $f \in \mathcal{L}^1(E)$ and (B.3.1) holds.

B.4 Sequences of Integrable Functions

The main theorem of this subsection is as follows.

Theorem 4.1. *(Dominated convergence, Lebesgue) Let f_m, $m = 1, 2, ...,$ be a sequence of measurable functions in E. Suppose that*
(i) $f_m \to f$ a.e. in E;
(ii) $|f_m| \leq F$ a.e. in E for all m and for some $F \in \mathcal{L}^1(E)$.
Then $f \in \mathcal{L}^1(E)$ and

$$\lim_{m\to\infty} \int_E f_m(x)dx = \int_E f(x)dx. \tag{B.4.2}$$

Theorem 4.2. *(Beppo Levi) Let $f_m \in \mathcal{L}^1(E)$, $m = 1, 2, ...,$ satisfying the conditions:*
(i) $\sup_m \int_E f_m dx < \infty$;
(ii) $f_m \leq f_{m+1}$ a.e. in E for any m.
Then
(i) there exists $f \in \mathcal{L}^1(E)$ such that $f_m \to f$ a.e. in E as $m \to \infty$;
(ii) (B.4.2) holds.

Lemma 4.3. *(Fatou's) Let $f_m \in \mathcal{L}^1(E)$ and $f_m \geq 0$ a.e. in E for all $m \in \mathcal{N}$. Let $f_m \to f$ a.e. in E and $\sup_m \int_E f_m dx \leq M < \infty$. Then $f \in \mathcal{L}^1(E)$ and $\int_E f dx \leq M$.*