C4.1 Further Functional Analysis – Problem Sheet 0

This problem sheet is not for handing in. It is intended for revision and consolidation, during the long vacation and the beginning of Week 1 of MT, of some important concepts in Functional Analysis. Solutions are provided overleaf.

1. Let X be a normed vector space. Prove that X is a Banach space if and only if every absolutely convergent series with terms in X converges to a limit in X.

2. Let X_n, $n \geq 1$, be normed vector spaces. Consider the vector space X of sequences $(x_n)_{n=1}^\infty$ such that $x_n \in X_n$, $n \geq 1$, and $\sum_{n=1}^\infty \|x_n\| < \infty$, endowed with the norm

$$\|x\| = \sum_{n=1}^{\infty} \|x_n\|, \quad x = (x_n)_{n=1}^\infty \in X.$$

   (a) Prove that if X_n is complete for each $n \geq 1$ then so is X.

   (b) Let X_n^* denote the dual space of X_n, $n \geq 1$. Show that the dual space X^* of X is isometrically isomorphic to the vector space Y of all sequences $(f_n)_{n=1}^\infty$ such that $f_n \in X_n^*$, $n \geq 1$, and $\sup_{n\geq 1} \|f_n\| < \infty$, endowed with the norm given by $\|f\| = \sup_{n\geq 1} \|f_n\|$, $f = (f_n)_{n=1}^\infty \in Y$.

3. Let X be a Banach space.

   (a) What does it mean to say that an operator $T \in \mathcal{B}(X)$ is *invertible*?

   (b) Suppose that $T \in \mathcal{B}(X)$ and that $\|T\| < 1$. Show that $I - T$ is invertible.

   (c) Let $S, T \in \mathcal{B}(X)$ and suppose that T is invertible and that $\|S\| < \|T^{-1}\|^{-1}$. Prove that $S + T$ is invertible and that

$$(S + T)^{-1} = \sum_{n=1}^{\infty} (-1)^n (T^{-1}S)^n T^{-1},$$

   where the series converges in the norm of $\mathcal{B}(X)$.

   (d) Deduce that the set of invertible operators is an open subset of $\mathcal{B}(X)$ and that the spectrum

$$\sigma(T) = \{\lambda \in \mathbb{F} : \lambda - T \text{ is not invertible}\}$$

   of any operator $T \in \mathcal{B}(X)$ is a compact subset of the field $\mathbb{F}$.

   (e) Given a non-empty compact subset K of $\mathbb{F}$, show that there exist a Banach space X and $T \in \mathcal{B}(X)$ such that $\sigma(T) = K$. What can you say if K is empty?

4. Let X be a Banach space, Y a normed vector space and let $T \in \mathcal{B}(X, Y)$.

   (a) Suppose there exist $\varepsilon \in (0, 1)$ and $M > 0$ such that $\operatorname{dist}(y, T(B_X^\circ(M))) < \varepsilon$ for all $y \in B_Y^\circ$. Prove that $B_Y^\circ \subseteq T(B_X^\circ(M(1-\varepsilon)^{-1}))$.

   (b) Deduce that if $T(B_X^\circ(M))$ contains a dense subset of B_Y° then $B_Y^\circ \subseteq T(B_X^\circ(M))$.

   (c) State the Baire Category Theorem and the Open Mapping Theorem, and show how the latter can be deduced from the former.

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Solutions

1. Suppose first that X is complete and let $x_n \in X$, $n \geq 1$, be such that $\sum_{n=1}^{\infty} \|x_n\| < \infty$. If we let $s_n = \sum_{k=1}^{n} x_k$, $n \geq 1$, then for $n \geq m \geq 1$ we have

$$\|s_n - s_m\| \leq \sum_{k=m+1}^{n} \|x_k\| \leq \sum_{k=m+1}^{\infty} \|x_k\| \to 0, \quad m \to \infty,$$

so the sequence $(s_n)_{n=1}^{\infty}$ is Cauchy and therefore convergent.

Conversely, suppose that every absolutely convergent series in X converges to a limit in X, and let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in X. Then we can find a subsequence $(x_{n_k})_{k=1}^{\infty}$ such that $\|x_{n_k} - x_{n_l}\| \leq 2^{-k}$ for $l \geq k \geq 1$. Let $y_1 = x_{n_1}$ and $y_k = x_{n_k} - x_{n_{k-1}}$, $k \geq 2$. Then the series $\sum_{k=1}^{\infty} y_k$ is absolutely convergent, and hence by assumption there exists $x \in X$ such that $\|y_1 + \cdots + y_k - x\| \to 0$ as $k \to \infty$. Since $y_1 + \cdots + y_k = x_{n_k}$, $k \geq 1$, it follows that the original sequence $(x_n)_{n=1}^{\infty}$ has a convergent subsequence. Recalling that any sequence which is Cauchy and has a convergent subsequence must be convergent,[1] we deduce that X is complete.

2. (a) Suppose that the spaces X_n, $n \geq 1$, are complete and let $(x^{(k)})_{k=1}^{\infty}$ be a Cauchy sequence in X, writing $x^{(k)} = (x_n^{(k)})_{k=1}^{\infty}$. Moreover let $\varepsilon > 0$. Then there exists $K \geq 1$ such that

$$\sum_{n=1}^{\infty} \|x_n^{(k)} - x_n^{(\ell)}\| = \|x^{(k)} - x^{(\ell)}\| < \varepsilon, \quad k, \ell \geq K.$$

In particular, for each fixed $n \geq 1$ the sequence $(x_n^{(k)})_{k=1}^{\infty}$ is Cauchy in X_n. Since each X_n is complete there exist $x_n \in X_n$, $n \geq 1$, such that $\|x_n^{(k)} - x_n\| \to 0$ as $k \to \infty$. Let $x = (x_n)_{n=1}^{\infty}$. For $k \geq K$ we have

$$\sum_{n=1}^{N} \|x_n^{(k)} - x_n\| = \lim_{\ell \to \infty} \sum_{n=1}^{N} \|x_n^{(k)} - x_n^{(\ell)}\| \leq \varepsilon, \quad N \geq 1.$$

Taking limits as $N \to \infty$,

$$\sum_{n=1}^{\infty} \|x_n^{(k)} - x_n\| \leq \epsilon.$$

So $x^{(k)} - x \in X$ for $k \geq K$ and in particular $x \in X$. We also obtain that $\|x^{(k)} - x\| \leq \varepsilon$ for $k \geq K$, so X is complete, as required.

(b) Let

$$(\Phi y)(x) = \sum_{n=1}^{\infty} f_n(x_n)$$

[1] by a 2ϵ argument, which I leave you to recall / fill in.

for $x = (x_n)_{n=1}^\infty \in X$ and $y = (f_n)_{n=1}^\infty \in Y$. Note first that the series on the right-hand side is (absolutely) convergent because

$$\sum_{n=1}^N |f_n(x_n)| \le \max_{1\le n\le N} \|f_n\| \sum_{n=1}^N \|x_n\| \le \|y\|\|x\|, \quad N \ge 1.$$

Moreover, letting $N \to \infty$ we see that $|(\Phi y)(x)| \le \|y\|\|x\|$ for $x \in X$, $y \in Y$. Thus for each fixed $y \in Y$, Φy defines a bounded linear[2] map $X \to \mathbb{F}$ with $\|\Phi y\| \le \|y\|$. Hence, Φ is a well-defined linear[3] map from Y into X^*.

Fix $y = (f_n)_{n=1}^\infty \in Y$. For $n \ge 1$ and $z \in X_n$ let $e_n(z) \in X$ denote the sequence with z in the n-th position and zeros elsewhere. Then for $n \ge 1$ and $z \in X_n$ we have $\|e_n(z)\| = \|z\|$ and $|(\Phi y)(e_n(z))| = |f_n(z)|$, and hence

$$\|\Phi y\| \ge \sup_{z \in B_{X_n}} |f_n(z)| = \|f_n\|, \quad n \ge 1.$$

It follows that $\|\Phi y\| \ge \|y\|$, so Φ is an isometry and in particular injective. It remains to show that Φ is surjective. Given $f \in X^*$ we may define the bounded linear functionals $f_n \in X_n^*$, $n \ge 1$, by $f_n(z) = f(e_n(z))$, $z \in X_n$. Then $\|f_n\| \le \|f\|$, $n \ge 1$, so the sequence $y = (f_n)_{n=1}^\infty$ lies in Y. Furthermore, given $x_n \in X_n$, $n \ge 1$, we have

$$(\Phi y)(x_1, \dots, x_N, 0, 0, \dots) = \sum_{n=1}^N f_n(x_n) = f(x_1, \dots, x_N, 0, 0, \dots), \quad N \ge 1,$$

so Φy agrees with f on the subspace Z of X consisting of all finitely supported sequences. Since Z is dense[4] in X it follows from continuity of Φy and f that $\Phi y = f$, so Φ is surjective, as required.

> What we've done here is form the ℓ^1-direct sum X of the sequence $(X_n)_{n=1}^\infty$ of Banach spaces, and the ℓ^∞ direct sum Y of the spaces $(X_n^*)_{n=1}^\infty$, and then show that X^* is canonically isometrically isomorphic to Y. The proof is essentially the same as the canonical isometric isomorphism between ℓ^1 and ℓ^∞ which you may well have seen previously.
>
> How would you define ℓ^p direct sums of Banach spaces, and the c_0 sum, and what would you expect the duals to be?

3. (a) An operator $T \in \mathcal{B}(X)$ is invertible if there exists an operator $S \in \mathcal{B}(X)$ such that $ST = TS = I$, the identity operator on X. In this case we write $S = T^{-1}$.
   (b) Since X is complete so is $\mathcal{B}(X)$. If $T \in \mathcal{B}(X)$ satisfies $\|T\| < 1$, then the series $\sum_{n=0}^\infty T^n$ is absolutely convergent and therefore convergent in $\mathcal{B}(X)$. Denote

[2] this is an easy check
[3] another easy check
[4] check this

the limit by S and let $S_n = I + \cdots + T^{n-1}$, $n \geq 1$. Then $(I-T)S_n = S_n(I-T) = I - T^n$, $n \geq 1$, and hence

$$\|S(I-T)-I\| \leq \|(S-S_n)(I-T)\| + \|T^n\| \leq \|S-S_n\|\|I-T\| + \|T\|^n, \quad n \geq 1.$$

Letting $n \to \infty$ we see that $S(I-T) = I$. Similarly $(I-T)S = I$, so $I-T$ is invertible with inverse S.

(c) If $S, T \in \mathcal{B}(X)$ and T is invertible, then $S+T = T(I+T^{-1}S)$. Let $Q = I+T^{-1}S$. If $\|S\| < \|T^{-1}\|^{-1}$, then $\|ST^{-1}\| < 1$ and part (b) gives that Q is invertible. If we let $R = Q^{-1}T^{-1}$, then $R \in \mathcal{B}(X)$ and $(S+T)R = R(S+T) = I$, so $S+T$ is invertible. The formula for $(S+T)^{-1}$ follows from the argument in part (b).

(d) It is clear from part (c) that the set of invertible operators is an open subset of $\mathcal{B}(X)$. If $\lambda \in \mathbb{F} \setminus \sigma(T)$, then $\lambda - T$ is invertible, and hence for $\mu \in \mathbb{F}$ such that $|\mu - \lambda| < \|(\lambda - T)^{-1}\|^{-1}$ the operator $\mu - T$ is invertible by part (c). Hence $\mathbb{F} \setminus \sigma(T)$ is open, so $\sigma(T)$ is closed. Since $\lambda - T = \lambda(I - \lambda^{-1}T)$ for $\lambda \neq 0$, it follows from part (b) that $\lambda - T$ is invertible when $|\lambda| > \|T\|$. Thus $\sigma(T) \subseteq \{\lambda \in \mathbb{F} : |\lambda| \leq \|T\|\}$. In particular, $\sigma(T)$ is bounded and hence compact.

(e) If K is non-empty, we may consider a dense subset K_0 of K which is at most countably infinite. Suppose that $K_0 = \{\lambda_n : n \in \mathbb{N}\}$, where the sequence $(\lambda_n)_{n=1}^\infty$ is eventually constant in case K_0 is finite. Now let $X = \ell^1$ and let $T\colon X \to X$ be given by $T((x_n)_{n=1}^\infty) = (\lambda_n x_n)_{n=1}^\infty$ for $(x_n)_{n=1}^\infty \in X$. Then[5] $T \in \mathcal{B}(X)$ and each λ_n, $n \geq 1$, is an eigenvalue of T, so $K_0 \subseteq \sigma(T)$. Since the spectrum is closed it follows that $K \subseteq \sigma(T)$. If $\lambda \in \mathbb{F} \setminus K$, then $\delta = \operatorname{dist}(\lambda, K) > 0$. Thus the sequence $((\lambda - \lambda_n)^{-1})_{n=1}^\infty$ is bounded, so similarly defines a bounded operator $S \in \mathcal{B}(X)$ by $S((x_n)_{n=1}^\infty) = ((\lambda - \lambda_n)^{-1} x_n)_{n=1}^\infty$. It's easy to check that S is the inverse of $\lambda - T$, so $\lambda \notin \sigma(T)$. Thus $\sigma(T) = K$.

If K is empty and $\mathbb{F} = \mathbb{C}$ then K cannot be the spectrum of any operator $T \in \mathcal{B}(X)$, since over the complex field the spectrum is always non-empty.[6] On the other hand, if $\mathbb{F} = \mathbb{R}$ we may consider $X = \mathbb{R}^2$ with the Euclidean norm, say, and $T(x, y) = (y, -x)$ for $(x, y) \in X$. Then X is a Banach space and $T \in \mathcal{B}(X)$. The characteristic polynomial of T is $c_T(\lambda) = \lambda^2 + 1$, so $\sigma(T) = \emptyset$.

> For many of us most of this exercise will be bookwork from an earlier course. However if not, spectral theory will not be a major part of this course only really appearing towards the end of section 12. The difference between the real and complex field in (e), is one of the main reasons why many mathematicians studying operators on Hilbert or Banach spaces, typically prefer to work with complex scalars. When one's just looking at Banach spaces, and not focusing on the operators between them, this matters less, and one often uses real scalars (having the slight advantage that one doesn't need to take real parts in the separation theorems).

[5] Notice that this works because the sequence $(\lambda_n)_{n=1}^\infty$ is bounded.

[6] This is a deep result using the Hahn-Banach theorem to obtain Banach space versions of results from complex analysis.

4. (a) Let $y \in B_Y^\circ$. We recursively define sequences $(x_n)_{n=1}^\infty$ in X and $(y_n)_{n=1}^\infty$ in Y as follows. Set $y_1 = y$ and let $x_1 \in B_X^\circ(M)$ be such that $\|Tx_1 - y_1\| < \varepsilon$. Supposing we have $x_n \in X$ and $y_n \in Y$ such that $\|y_n\| < \varepsilon^{n-1}$, $\|x_n\| < M\varepsilon^{n-1}$ and $\|Tx_n - y_n\| < \varepsilon^n$, we set $y_{n+1} = y_n - Tx_n$. Since $\varepsilon^{-n}\|y_{n+1}\| < 1$ there exists $x'_{n+1} \in B_X^\circ(M)$ such that $\|Tx'_{n+1} - \varepsilon^{-n}y_{n+1}\| < \varepsilon$. If we let $x_{n+1} = \varepsilon^n x'_{n+1}$ then $\|x_{n+1}\| < M\varepsilon^n$ and we may continue inductively. Since $\sum_{n=1}^\infty \|x_n\| < \infty$ and X is complete, the series $\sum_{n=1}^\infty x_n$ converges to some $x \in X$ satisfying

$$\|x\| \le \sum_{n=1}^\infty \|x_n\| < \frac{M}{1-\varepsilon}.$$

Moreover

$$\left\| y - \sum_{k=1}^n Tx_k \right\| = \|y_{n+1}\| < \varepsilon^n \to 0, \quad n \to \infty.$$

By continuity of T we obtain that $Tx = y$.

(b) If $T(B_X^\circ(M))$ contains a dense subset of B_Y°, then $B_Y^\circ \subseteq T(B_X^\circ(M)) + B_Y^\circ(\varepsilon)$ and hence by the first part $B_Y^\circ(1-\varepsilon) \subseteq T(B_X^\circ(M))$ for all $\varepsilon \in (0,1)$. It follows that

$$B_Y^\circ = \bigcup_{\varepsilon \in (0,1)} B_Y^\circ(1-\varepsilon) \subseteq T(B_X^\circ(M)).$$

(c) **Baire Category Theorem.** *Let (X,d) be a complete metric space and suppose that U_n, $n \in \mathbb{N}$, are dense open subsets of X. Then $\bigcap_{n\ge1} U_n$ is dense in X.*

Open Mapping Theorem. *Let X and Y be Banach spaces and suppose that $T \in \mathcal{B}(X,Y)$ is a surjection. Then T is an open map.*

Suppose that X, Y are Banach spaces and that $T \in \mathcal{B}(X,Y)$ is surjective. Then $Y = \bigcup_{n=1}^\infty F_n$, where F_n denotes the closure of $T(B_X(n))$, $n \ge 1$. Let $U_n = Y \setminus F_n$, $n \ge 1$. Then each U_n, $n \ge 1$, is open and $\bigcap_{n\ge1} U_n = \emptyset$, so by the Baire Category Theorem there exists $k \ge 1$ such that U_k fails to be dense in Y. Hence F_k has non-empty interior, so $F_k \supseteq B_Y^\circ(y,\varepsilon)$ for some $y \in Y$ and $\varepsilon > 0$. By symmetry and convexity $B_Y^\circ(\varepsilon) \subseteq F_k$, so part (b) gives $B_Y^\circ \subseteq T(B_X^\circ(M))$ for $M = \varepsilon^{-1}k$. It follows using linearity that T is an open map.

> Part (a) is the successive approximation lemma; it'll appear at the end of section 4 of the lecture notes. This trick of repeatedly approximating to obtain an exact solution is a useful idea in analysis, and is well worth filing away.