C4.1 Further Functional Analysis – Problem Sheet 1

For classes in Week 2/3 of MT

1. (a) Let X, Y and Z be vector spaces and suppose that $T\colon X \to Y$ and $S\colon X \to Z$ are linear maps. Show that there exists a linear map $\pi\colon Z \to Y$ such that $T = \pi \circ S$ if and only if $\operatorname{Ker} S \subseteq \operatorname{Ker} T$.

   (b) Hence or otherwise show that if $n \in \mathbb{N}$ and if $f_1, \dots, f_n$ and f are linear functionals on a vector space X, then $f \in \operatorname{span}\{f_1, \dots, f_n\}$ if and only if

$$\bigcap_{k=1}^{n} \operatorname{Ker} f_k \subseteq \operatorname{Ker} f.$$

   What can you deduce about double annihilators of finite-dimensional subspaces of X^* when X is a normed vector space?[1]

2. Let X be an infinite-dimensional normed space, and suppose that $\{x_\alpha : \alpha \in A\}$ is a Hamel basis for X and that $\|x_\alpha\| = 1$ for all $\alpha \in A$. Given a vector $x \in X$ which has the expansion $x = \sum_{\alpha \in A} \lambda_\alpha x_\alpha$ we let

$$|||x||| = \sum_{\alpha \in A} |\lambda_\alpha|.$$

   (a) Check that $||| \cdot |||$ defines a norm on X.

   (b) Now let X be a Banach space. Show that $(X, ||| \cdot |||)$ is not separable.

   (c) Deduce that in the Closed Graph Theorem the assumption that the codomain be complete cannot be omitted.

3. Let X be a vector space on which two norms $\| \cdot \|, ||| \cdot |||$ are defined, and suppose that $\|x\| \le C|||x|||$ for some constant $C > 0$ and all $x \in X$.

   (a) Show that if X is complete with respect to one of the two norms then it is complete with respect to the other if and only if the two norms are equivalent.

   (b) Give an example in which $(X, ||| \cdot |||)$ is complete but $(X, \| \cdot \|)$ is not.

   (c) Give an example in which $(X, \| \cdot \|)$ is complete but $(X, ||| \cdot |||)$ is not.

4. Let X be an infinite-dimensional Banach space with norm $\| \cdot \|$, and let $f\colon X \to \mathbb{F}$ be an unbounded linear functional. Given a vector $x_0 \in X$ such that $f(x_0) = 1$, consider the linear operator $T\colon X \to X$ defined by

$$Tx = x - 2f(x)x_0, \quad x \in X.$$

   Show that $T^2 = I$. Hence show that the map $||| \cdot |||\colon X \to [0, \infty)$ given, for $x \in X$, by $|||x||| = \|Tx\|$ defines a complete norm on X which is not equivalent to $\| \cdot \|$.

5. Let X be a vector space and suppose that Y is a subspace of X.

[1] We won't have covered annihilators yet, but you won't need much more than the definitions (found just before Corollary 5.8) for this question.

(a) Construct a linear map $P\colon X \to X$ such that $P^2 = P$ and $\operatorname{Ran} P = Y$.

(b) Deduce that Y is *algebraically complemented* in X, which is to say that there exists a further subspace Z of X such that every $x \in X$ can be expressed uniquely as $x = y + z$ with $y \in Y$ and $z \in Z$.

(c) Is the subspace Z in part (b) uniquely determined by Y?

6. Let X be a normed vector space and let Y be a subspace of X.

   (a) Suppose that Y is finite-dimensional. Show that Y is complemented in X, and that if Z is any closed subspace of X such that $X = Y \oplus Z$ algebraically, then X is in fact the topological direct sum of Y and Z.

   (b) What can you say if Y has finite codimension in X? [*Recall that the codimension of Y in X is the dimension of the quotient vector space X/Y.*]

7. Let $Y, Z \subseteq \ell^2$ be given by

$$Y = \big\{(y_n) \in \ell^2 : y_{2n} = 0 \text{ for all } n \geq 1\big\},$$
$$Z = \big\{(z_n) \in \ell^2 : z_{2n-1} = n z_{2n} \text{ for all } n \geq 1\big\}.$$

   (a) Show that Y and Z are closed subspaces of ℓ^2 and that $Y \cap Z = \{0\}$.

   (b) Letting $X = Y \oplus Z$ denote the algebraic direct sum of Y and Z, prove that X is dense in ℓ^2 but that $X \neq \ell^2$, and deduce that X is not the topological direct sum of Y and Z.

   (c) Let $P\colon X \to X$ be the linear map given by $P(y+z) = y$ for all $y \in Y$, $z \in Z$. Show directly that P is unbounded.

8. Let X be a normed vector space and let Y and Z be subspaces of X such that $X = Y \oplus Z$ algebraically. Show that if Y is closed, then X is the topological direct sum of Y and Z if and only if the restriction $\pi|_Z\colon Z \to X/Y$ of the canonical quotient map $\pi\colon X \to X/Y$ is an isomorphism.

9. Let Y and Z be closed subspaces of a Banach space X with $Y \cap Z = \{0\}$. Equip the algebraic direct sum $Y \oplus Z$ with the ℓ^1-norm: $|||y + z||| = \|y\| + \|z\|$.

   (a) Show that $|||\cdot|||$ is complete on $Y \oplus Z$.

   (b) Show that the following are equivalent:

      i. $|||\cdot|||$ is equivalent to the original norm on $Y \oplus Z$ (as a subspace of X);

      ii. $Y \oplus Z$ is closed in X;

      iii. Y is complemented by Z in $Y + Z$ (so $Y \oplus Z$ is a topological direct sum).[2]

SAW MT19

[2]Insertion 'by Z' added 22 Oct 19; thanks to the student who pointed this out. As an extra exercise, why does this matter? Can Y be complemented by something other than Z?