C4.1 Further Functional Analysis – Problem Sheet 2

For classes in Weeks 4 and 5 of MT

1. (a) Let X be a Banach space and suppose that $\{x_n : n \geq 1\}$ is a bounded subset of X. Show that there exists a unique operator $T \in \mathcal{B}(\ell^1, X)$ such that $Te_n = x_n$ for all $n \geq 1$ and $\|T\| = \sup_{n\geq 1} \|x_n\|$.
   (b) Prove that if X is a separable Banach space then $X \cong \ell^1/Y$ for some closed subspace Y of ℓ^1.
   (c) Deduce that ℓ^1 contains closed subspaces which are uncomplemented. [*You may assume that any closed infinite-dimensional subspace of ℓ^1 has non-separable dual. We shall prove this at the end of the course.*]

2. (a) Prove that the Closed Graph Theorem, the Inverse Mapping Theorem and the Open Mapping Theorem are all equivalent.
   (b) Let X and Y be Banach spaces and suppose that $T \in \mathcal{B}(X, Y)$ is such that $\operatorname{Ran} T$ has finite codimension in Y. Show that $\operatorname{Ran} T$ is closed.

3. (a) Let X be an infinite dimensional real normed space, and $f : X \to \mathbb{R}$ a linear functional. Show that if there is an open ball $B^0_X(x_0, r)$ such that $f(x) > 0$ for $x \in B^0_X(x_0, r)$, then f is continuous. Deduce that if f is unbounded, then $\ker f$ is dense in X.
   (b) Use the previous result to show that any infinite dimensional normed space X can be decomposed into a union $A \cup B$ of disjoint convex sets, with both A and B dense in X.

4. (a) Let C be a convex absorbing subset of a normed space. Show

$$\{x \in X : p_C(x) < 1\} \subseteq C \subseteq \{x \in X : p_C(x) \leq 1\},$$

   with equality in the first inclusion when C is open, and equality in the second when C is closed.
   (b) Let C be a convex balanced subset of a normed space, which contains a neighbourhood of 0 and is bounded. Show that p_C gives an equivalent norm on X.
   (c) Let Y be a subspace of a normed space $(X, \cdot)$, and let $|||\cdot|||$ be an equivalent norm on Y. Show that $|||\cdot|||$ can be extended to an equivalent norm on X.

5. Let X and Y be normed vector spaces and let $T \in \mathcal{B}(X, Y)$. Suppose there exists a constant $r > 0$ such that $\|T^*f\| \geq r\|f\|$ for all $f \in Y^*$.
   (a) Using the Hahn-Banach Separation Theorem, or otherwise, show that $B_Y(r)$ is contained in the closure of $T(B_X)$.
   (b) If X is complete, deduce that T is a quotient operator, and that T is an isometric quotient operator if T^* is an isometry.

6. (a) * Let X be a normed vector space and let $P \in \mathcal{B}(X^{***})$ be given by $P = J_{X^*}J_X^*$. Show that P is the projection onto $J_{X^*}(X^*)$ along $J_X(X)^\circ$ and that $\|P\| = 1$.

(b) (i) Show that if $T \in \mathcal{B}(\ell^\infty)$ with $\|T\| = 1$ and $Te_n = e_n$, $n \geq 1$, then $T = I$.

(ii) Deduce that there does not exist a projection of norm 1 from ℓ^∞ onto c_0.

(iii) Prove that there is no normed vector space X such that $X^* \cong c_0$.

7. Let $X = \ell^\infty$ and $Sx = (x_{n+1})$ for $x = (x_n) \in X$. Moreover, let $T = I - S$.

   (a) Show that $\operatorname{Ker} T = \{(\lambda, \lambda, \lambda, \dots) : \lambda \in \mathbb{F}\}$ and that $\operatorname{Ran} T \cap \operatorname{Ker} T = \{0\}$.

   (b) Let $Y = \operatorname{Ran} T \oplus \operatorname{Ker} T$ and let $P \colon Y \to Y$ be the projection onto $\operatorname{Ker} T$ along $\operatorname{Ran} T$. By considering the operators

   $$A_n = \frac{1}{n} \sum_{k=0}^{n-1} S^k, \quad n \geq 1,$$

   or otherwise, show that P is bounded and that $\|P\| = 1$.

   (c) Prove that there exists a functional $f \in X^*$ with $\|f\| = 1$ such that $f(Sx) = f(x)$ for all $x \in X$ and

   $$f(x) = \lim_{n \to \infty} x_n$$

   whenever $x = (x_n) \in c$.[1] Evaluate $f(x)$ when x is a periodic sequence.

8. Given a normed vector space X, we say that X is injective [2] if whenever Y is a subspace of a normed vector space Z and $T \in \mathcal{B}(Y, X)$ there exists an operator $S \in \mathcal{B}(Z, X)$ such that $\|S\| = \|T\|$ and $S|_Y = T$.

   (a) (i) Show that ℓ^∞ is injective

   (ii) By proving first that any operator $T \in \mathcal{B}(\ell^\infty, c_0)$ such that $Te_n = e_n$, $n \geq 1$, must have norm $\|T\| \geq 2$, or otherwise, show that c_0 is not injective.

   (iii) * Is c_0 complemented in c, and if so what can you say about the norms of a complementing projection?

   (b) Suppose that X is an injective normed vector space, and Y is a subspace of a normed vector space Z such that Y is isomorphic to X. Prove that Y is complemented in Z.

9. Let X be a normed vector space and let Y be a subspace of X.

   (a) Writing $Y^{\circ\circ} = (Y^\circ)^\circ$ for the double annihilator of Y in X^{**}, show that there exists an isometric isomorphism $T \colon Y^{**} \to Y^{\circ\circ}$ such that $T \circ J_Y = J_X|_Y$.

   (b) Show that if X is reflexive and Y is closed, then both Y and X/Y are reflexive.

SAW MT19

[1] Recall that c is the subspace of ℓ^∞ consisting of convergent sequences.

[2] The terminology comes from category theory; X is an injective object in the category of normed spaces with contractive linear maps.