C4.1 Further Functional Analysis – Problem Sheet 4

For classes in Week 1 of HT.

1. (a) Let X and Y be normed vector spaces and suppose that $T_n \in \mathcal{B}(X,Y)$, $n \geq 1$, satisfy $\sup_{n\geq 1} \|T_n\| < \infty$. Let M be a totally bounded subset of X such that

$$\lim_{n\to\infty} \|T_n x\| = 0, \quad x \in M.$$

   Prove that the convergence is uniform over $x \in M$.

   (b) Let X be a Banach space and suppose that there exist finite-rank operators $P_n \in \mathcal{B}(X)$, $n \geq 1$, such that

$$\lim_{n\to\infty} \|P_n x - x\| = 0, \quad x \in X.$$

   (i) Show that a subset M of X is totally bounded if and only if it is bounded and $\|P_n x - x\| \to 0$ as $n \to \infty$ uniformly over $x \in M$.

   (ii) Prove that a bounded linear operator on X is compact if and only if it is the norm limit of a sequence of finite-rank operators.

   Note that (b)(ii) shows that if X is a Banach space with a Schauder basis, then compact operators on X are precisely the norm limits of finite-rank operators. In particular, this holds when X is a separable Hilbert space.[1]

2. Let $K \in L^2(\mathbb{R}^2)$ and consider the map T sending $x \in L^2(\mathbb{R})$ to the function Tx defined by

$$(Tx)(t) = \int_{\mathbb{R}} K(s,t)x(s)\,\mathrm{d}s$$

   whenever $t \in \mathbb{R}$ is such that the integral exists.

   (a) Show that T is a well-defined element of $\mathcal{B}(L^2(\mathbb{R}))$ with $\|T\| \leq \|K\|_{L^2(\mathbb{R}^2)}$.

   (b) Prove that T is compact. [*You may use the fact that indicator functions of bounded rectangles span a dense subspace of* $L^2(\mathbb{R}^2)$.]

3. Let X and Y be normed vector spaces and let $T \in \mathcal{K}(X,Y)$. Furthermore, let Ω denote the closure of $T(B_X)$ and let $M = \{f|_\Omega : f \in B_{Y^*}\}$.

   (a) Prove that M is a relatively compact subset of $C(\Omega)$.

   (b) Show that M and $T^*(B_{Y^*})$ are isometric.

   (c) Deduce Schauder's Theorem.

4. (a) Let X and Y be normed vector spaces and let $T \in \mathcal{B}(X,Y)$. We say that T is *completely continuous* if, for every weakly convergent sequence (x_n) in X, the sequence (Tx_n) is norm-convergent in Y.

   (i) Show that if T is compact then T is completely continuous.

   (ii) Prove that the converse of (i) holds if X is reflexive. [*You may, if you wish, assume in addition that X is separable.*]

[1] Additional exercise. Show that regardless of separability, every compact operator on a Hilbert space is a limit of finite rank operators.

(iii) Exhibit an operator which is completely continuous but not compact.

(b) Let $1 < p < \infty$. Show that $\mathcal{B}(\ell^p, \ell^1) = \mathcal{K}(\ell^p, \ell^1)$. Is $\mathcal{B}(c_0, \ell^p) = \mathcal{K}(c_0, \ell^p)$?

5. Let X and Y be normed vector spaces and let $T \in \mathcal{B}(X, Y)$. We say that T is *weakly compact* if the weak closure of $T(B_X)$ is weakly compact.

   (a) Show that T is weakly compact if and only if $\operatorname{Ran} T^{**} \subseteq J_Y(Y)$.

   (b) Prove that if T is weakly compact then T^* is weakly compact, and that if Y is complete then the converse holds too.

6. Let X, Y be Banach spaces and suppose that $T \in \mathcal{B}(X, Y)$. Show that T is Fredholm if and only if T^* is and that, if both operators are Fredholm, then $\operatorname{ind} T + \operatorname{ind} T^* = 0$.

7. Let X, Y and Z be Banach spaces and let $S \in \mathcal{B}(Y, Z)$ and $T \in \mathcal{B}(X, Y)$.

   (a) Show that if S, T are both Fredholm then so is ST and $\operatorname{ind} ST = \operatorname{ind} S + \operatorname{ind} T$.

   (b) Suppose now that ST is Fredholm. Prove that S is Fredholm if and only if T is Fredholm. Give an example in which neither S nor T is Fredholm.

   (c) Show that if $X = Y = Z$ and $ST = TS$ then ST is Fredholm if and only if S and T are both Fredholm.

8. Let X be the complex Banach space ℓ^1 and consider the left-shift operator $T \in \mathcal{B}(X)$ given by $Tx = (x_{n+1})_{n\geq 1}$ for $x = (x_n)_{n\geq 1} \in X$. Moreover let $\Gamma = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

   (a) Show that for $\lambda \in \mathbb{C}$ the operator $T - \lambda$ is Fredholm if and only if $\lambda \notin \Gamma$, and determine the index $\operatorname{ind}(T - \lambda)$ whenever it is defined.

   (b) Let p be a complex polynomial. Prove that $p(T)$ is Fredholm if and only if $p^{-1}(\{0\}) \cap \Gamma = \emptyset$ and that, if this condition is satisfied, then

$$\operatorname{ind} p(T) = \frac{1}{2\pi i} \oint_\Gamma \frac{p'(\lambda)}{p(\lambda)} \, \mathrm{d}\lambda.$$

9. Let X be a Banach space and let $\{x_n : n \geq 1\}$ be a Schauder basis for X with basis projections P_n, $n \geq 1$, and let

$$|||x||| = \sup\{\|P_n x\| : n \geq 1\}, \quad x \in X.$$

   Prove that $|||\cdot|||$ defines a complete norm on X.

10. (a) Let X be a Banach space and suppose that $\{x_n : n \geq 1\} \subseteq X \setminus \{0\}$ spans a dense subspace of X. Prove that $\{x_n : n \geq 1\}$ is a Schauder basis for X if and only if there exists a constant $M > 0$ such that

$$\left\| \sum_{k=1}^{m} \lambda_k x_k \right\| \leq M \left\| \sum_{k=1}^{n} \lambda_k x_k \right\|$$

   for all $n \geq m \geq 1$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$.

   (b) Let $x_n = (1, 1, \ldots, 1, 0, 0 \ldots)$, $n \geq 1$, where the last 1 appears in the n-th entry. Prove that $\{x_n : n \geq 1\}$ is a Schauder basis for c_0.

(c) Let X be a Banach space which admits a Schauder basis $\{x_n : n \geq 1\}$ with associated basis functionals $f_n \in X^*$, $n \geq 1$.

  (i) Show that the set $\{f_n : n \geq 1\}$ is *basic*, which is to say that it forms a Schauder basis for its closed linear span.

  (ii) Assuming that X^* admits a Schauder basis, is $\{f_n : n \geq 1\}$ necessarily a Schauder basis for X^*?

This ends the core problem sheet. Some extensional problems are below, but these are not for handing in, and are unlikely to be discussed in class. If there's sufficient demand, I'll be happy to arrange to discuss them.

11. Let X be a Banach space with a Schauder basis $\{x_n : n \geq 1\}$ with associated basis projections P_n and basis functionals $f_n \in X^*$, $n \geq 1$.

    (a) Somewhat giving away the answer to the last part of the previous question, show that for each $n \in \mathbb{N}$,

    $$\|f|_{\overline{\text{Span}\{x_i : i > n\}}} \leq \|f - P_n^* f\| \leq (1 + K)\|f|_{\overline{\text{Span}\{x_i : i > n\}}},$$

    where K is the basis constant. Deduce that $\overline{\text{Span}}\{f_n : n \in \mathbb{N}\} = X^*$ if and only if for every $f \in X^*$,

    $$\|f|_{\overline{\text{Span}\{e_i : i > n\}}}\| \to 0,$$

    as $n \to \infty$. [In this case we say that $\{x_n : n \geq 1\}$ is a *shrinking Schauder basis*.].

    (b) Let $f \in X^*$. Observe that $P_n^* f \to f$ weak*, and use this to deduce that if X is reflexive, then $\{x_n : n \geq 1\}$ is shrinking.

    (c) Suppose that $\{x_i : i \in \mathbb{N}\}$ is shrinking. Let Y be the space of all sequences $(a_n)_{n=1}^\infty$ equipped with $\|(a_n)\| = \sup_n \|\sum_{i=1}^n a_i x_i\|$. Verify that this is a norm on Y, and that $T : X^{**} \to Y$ given by $(T\phi) = (\phi(f_n)_{n=1}^\infty)$ is an isomorphism. Show too that if the basis constant is 1, then T is isometric. [For context, think about what is going on with the canonical basis of c_0.].

    (d) Deduce that X is a reflexive space if and only if $\{x_i : i \in \mathbb{N}$ is shrinking and for all sequence of scalars $(a_n)_{n=1}^\infty$, $\sum_{i=1}^\infty a_i x_i$ converges whenever $\sup_n \|\sum_{i=1}^n a_i x_i\| < \infty$.

12. The *James space* is the space X consisting of all sequences of real numbers $(a_n)_{n=1}^\infty$ such that $a_n \to 0$ and

    $$\|(a_n)\| = \sup_{k \geq 2} \sup_{n_1 < n_2 < \cdots < n_k} \Big(\sum_{i=1}^{k-1} (a_{n_i} - a_{n_{i+1}})^2\Big)^{1/2} < \infty.$$

    (a) Show that X is a Banach space, and that the elements e_n (which have a 1 in the n-th position and zeros elsewhere) form a Schauder basis for X with basis constant 1.

(b) Suppose, with the aim of reaching a contradiction, that $\{e_n : n \in \mathbb{N}\}$ is not shrinking in the sense of the previous question. Use 11(a), to find $f \in X^*$ with $\|f\| = 1$, $\epsilon > 0$, a real sequence $(a_n)_{n=1}^{\infty}$ and $p_1 < q_1 < p_2 < q_2 < \dots$ such that the elements $x_n = \sum_{i=p_n}^{q_n} a_i e_i \in X$ have $\|x_n\| = 1$ and $f(x_n) > \epsilon$ for all n. By considering

$$b_n = \begin{cases} a_n/n, & p_n \leq n \leq q_n \\ 0, & \text{otherwise} \end{cases},$$

or otherwise reach a contradiction, and deduce that $\{e_n : n \in \mathbb{N}\}$ is shrinking.

(c) Given a real sequence (a_n) such that $\sup_n \|\sum_{i=1}^{n} a_i e_i\| < \infty$, show that $\lim_{n\to\infty} a_n$ exists. Use question 11 to deduce that $J_X(X)$ is has co-dimension 1 in X^{**}.

(d) Show that X is isomorphic to X^{**}.

(e) In Q10(c)(ii), your example probably had the property the dual space was not separable. Can you now give an example with a separable dual space?

13. (a) We first deal with rearrangements of series in Banach spaces. Say that a series $\sum_{n=1}^{\infty} x_n$ in a normed space X *unconditionally converges* to $x \in X$, if for every $\epsilon > 0$, there exists a finite set $F \subset \mathbb{N}$ such that for any finite set $G \subset \mathbb{N}$ with $G \supset F$, we have $\|\sum_{n\in G} x_n - x\| < \epsilon$.

   i. Let X be a normed space. Show that $\sum_{n=1}^{\infty} x_n$ unconditionally converges to $x \in X$ if and only if given any permutation θ of $\mathbb{N}$, we have $\sum_{n=1}^{\infty} x_{\theta(n)} = x$.

   ii. Say that a series $\sum_{n=1}^{\infty} x_n$ is *unconditionally Cauchy* if, for every $\epsilon > 0$, there exists a finite set $F \subset \mathbb{N}$, such that for any finite subset $F' \subset \mathbb{N}$ with $F \cap F' = \emptyset$, $\|\sum_{n\in F'} x_n\| < \epsilon$. Show that in a Banach space a series is unconditionally convergent if and only if it is unconditionally Cauchy.

   iii. Give an example of a unconditionally convergent series in a Banach space which is not absolutely convergent.

   iv. Let X be a real Banach space. Show that $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent if and only if for all sequences $(\epsilon_n)_{n=1}^{\infty}$ from $\{-1, +1\}$, the series $\sum_{n=1}^{\infty} \epsilon_n x_n$ converges.

(b) We now return to bases. Let X be a real Banach space. Say that a Schauder basis $\{x_n : n \in \mathbb{N}\}$ is *unconditional* if for every $x \in X$, the expression $x = \sum_{n=1}^{\infty} a_n x_n$ is unconditionally convergent. We say that a sequence $(x_n)_{n=1}^{\infty}$ in X is an unconditional basic sequence if $\{x_n : n \in \mathbb{N}\}$ is an unconditional Schauder basis for $\mathrm{Span}\{x_n : n \in \mathbb{N}\}$.

   i. Let $\{x_n : n \in \mathbb{N}\}$ be an unconditional Schauder basis for X. Show that there is a constant $K > 0$ such that whenever $(a_n)_{n=1}^{\infty}$ is a sequence of scalars such that $\sum_{n=1}^{\infty} a_n x_n$ converges, then

$$\|\sum_{n=1}^{\infty} \lambda_n a_n x_n\| \leq K \sup_n |\lambda_n| \|\sum_{n=1}^{\infty} a_n x_n\|$$

for all $(\lambda_n)_{n=1}^{\infty} \in \ell^{\infty}$.

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