# Analytic Number Theory Revision Class

- There will be 4 sessions each 1 hour.
  (2 by me, 2 by Zoe).

- Please look at 'Course Materials' page for C3.8.
  - comments on past exams.
  - document on content of revision classes.

- Please email requests <u>4 days</u> before the class
  will publicise main content to be covered ~3 days beforehand.

Next class: Friday 15th May. Requests by end of <u>today</u>!
(Requests can be on anything - exam Q's, example sheets, lecture notes etc.)

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Today: 2014/2015/2016 Q1 - multiplicative functions.

Question: Do you want us to cover/leave 2019 questions?

**SECOND PUBLIC EXAMINATION**

**Honour School of Mathematics Part C: Paper C3.8**
**Honour School of Mathematic and Computer Science Part C: Paper C3.8**
**Honour School of Mathematics and Philosophy Part C: Paper C3.8**

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# ANALYTIC NUMBER THEORY

## Trinity Term 2016

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**TUESDAY, 7 JUNE 2016, 2.30pm to 4.00pm**

*You may submit answers to as many questions as you wish but only the best two will count for the total mark.*

*You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.*

**Do not turn this page until you are told that you may do so**

1. (a) [12 marks]

What does it mean to say that $f(n)$ is a multiplicative arithmetic function?

Suppose $f(n)$ is multiplicative and that the Dirichlet series $F(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$ has $\sigma_0$ as its abscissa of absolute convergence. State and prove the Euler product formula for $F(s)$.

(b) [8 marks]

If $p_1, \ldots, p_k$ are distinct primes, and $e_1, \ldots, e_k$ are non-negative integers, write down a formula for

$$\sigma\left(p_1^{e_1} \ldots p_k^{e_k}\right),$$

where $\sigma(n)$ is the sum of the divisors of $n$, as usual.

This should be 3 not 2!

Using the Euler product formula, show that

$$\sum_{n=1}^{\infty} \frac{\sigma(n^2)}{n^s} = \frac{\zeta(s)\zeta(s-1)\zeta(s-2)}{\zeta(2s-2)}$$

for $\Re(s) > 2$.

[*You may assume without proof that the series on the left has* $\sigma_0 = 2$ *as its abscissa of absolute convergence.*]

(c) [5 marks]

Define the function $\Omega(n)$ by

$$\Omega\left(p_1^{e_1} \ldots p_k^{e_k}\right) = e_1 + \ldots + e_k.$$

If $m \in \mathbb{N}$, write down the Euler product for the function

$$F^{(m)}(s) = \sum_{n=1}^{\infty} m^{\Omega(n)} n^{-s},$$

and show that it has abscissa of absolute convergence greater than or equal to $(\log m)/(\log 2)$.

[*You may assume without proof that the function* $f(n) = k^{\Omega(n)}$ *is multiplicative.*]

1. (a) [12 marks]

What does it mean to say that $f(n)$ is a multiplicative arithmetic function?

Suppose $f(n)$ is multiplicative and that the Dirichlet series $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ has $\sigma_0$ as its abscissa of absolute convergence. State and prove the Euler product formula for $F(s)$.

- Bookwork.

$f(ab) = f(a)f(b)$ whenever $(a,b)=1$.

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 + \sum_{j=1}^{\infty} \frac{f(p^j)}{p^{js}}\right)$$

'abscissa of absolute convergence' = smallest $\sigma$ s.t. sum converges absolutely for $Re(s) > \sigma$.

(b) [8 marks]

If $p_1, \ldots, p_k$ are distinct primes, and $e_1, \ldots, e_k$ are non-negative integers, write down a formula for

$$\sigma\left(p_1^{e_1} \ldots p_k^{e_k}\right),$$

where $\sigma(n)$ is the sum of the divisors of $n$, as usual.

Using the Euler product formula, show that

$$\sum_{n=1}^{\infty} \frac{\sigma(n^2)}{n^s} = \frac{\zeta(s)\zeta(s-1)\zeta(s-2)}{\zeta(2s-2)}$$

for $\Re(s) > 2$.

[*You may assume without proof that the series on the left has* $\sigma_0 = 2$ *as its abscissa of absolute convergence.*]

$\sigma$ is multiplicative: If $(a,b)=1$

$$\sigma(a)\sigma(b) = \sum_{\substack{d|a \\ e|b}} de = \sum_{f|ab} f = \sigma(f)$$

since there is a bijection $f|ab \to \begin{matrix} (f,a)\,|\,a \\ (f,b)\,|\,b \end{matrix}$

(since $(a,b)=1$). $\qquad de|ab \leftarrow \begin{matrix} d|a \\ e|b \end{matrix}$

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$$\sigma(p_1^{e_1} \ldots p_k^{e_k}) = \prod_{i=1}^{k} \sigma(p_i^{e_i}) \quad (\text{since mult.})$$

but $\sigma(p_i^{e_i}) = 1 + p_i + p_i^2 + \ldots + p_i^{e_i} = \dfrac{p_i^{e_i+1} - 1}{p_i - 1}$.

$$\therefore \sigma(p_1^{e_1} \ldots p_k^{e_k}) = \prod_{i=1}^{k} \frac{p_i^{e_i+1} - 1}{p_i - 1}.$$

$\sigma(n)$ is mult. $\Rightarrow \sigma(n^2)$ mult.

$$\therefore \quad \sum \frac{\sigma(n^2)}{n^s} = \prod_p \left( \sum_{j=0}^{\infty} \frac{p^{2j+1}-1}{p-1} p^{-js} \right)$$

$$\left(\text{for } Re(s)>2\right) \quad = \prod_p \left( \frac{1}{p-1} \sum_{j=0}^{\infty} \left( p^{(2-s)j+1} - p^{-js} \right) \right)$$

$$= \prod_p \left( \frac{1}{p-1} \left( \frac{p}{1-p^{2-s}} - \frac{1}{1-p^{-s}} \right) \right)$$

$$= \prod_p \frac{\left(p - p^{1-s} - 1 + p^{2-s}\right)}{(1-p^{-s})(1-p^{2-s})(p-1)}$$

$$\frac{\zeta(s)\zeta(s-1)\zeta(s-2)}{\zeta(2s-2)} = \prod_p \frac{1-p^{2-2s}}{(1-p^{-s})(1-p^{1-s})(1-p^{2-s})}$$

(c) [5 marks]
Define the function $\Omega(n)$ by

$$\Omega\left(p_1^{e_1} \dots p_k^{e_k}\right) = e_1 + \dots + e_k.$$

If $m \in \mathbb{N}$, write down the Euler product for the function

$$F^{(m)}(s) = \sum_{n=1}^{\infty} m^{\Omega(n)} n^{-s},$$

and show that it has abscissa of absolute convergence greater than or equal to $(\log m)/(\log 2)$.
[*You may assume without proof that the function $f(n) = k^{\Omega(n)}$ is multiplicative.*]

$m^{\Omega(n)}$ is mult.

$\therefore$ in region of absolute convergence:

$$F^{(m)}(s) = \prod_p \left( \sum_{j=0}^{\infty} \frac{m^{\Omega(p^j)}}{p^{js}} \right)$$

$$= \prod_p \left( \sum_{j=0}^{\infty} \left( \frac{m}{p^s} \right)^j \right)$$

$$= \prod_p \left( 1 - \frac{m}{p^s} \right)^{-1}$$

If $n = 2^e$ then $\left| \frac{m^{\Omega(n)}}{n^s} \right| = \left| \frac{m^e}{2^{es}} \right| \geq 1$ if $\mathrm{Re}(s) \leq \frac{\log m}{\log 2}$.

So $\sum_{n \geq 1} \frac{m^{\Omega(n)}}{n^s}$ can't converge absolutely for $\mathrm{Re}(s) \leq \frac{\log m}{\log 2}$.

**SECOND PUBLIC EXAMINATION**

**Honour School of Mathematics Part C: Paper C3.8**
**Honour School of Mathematic and Computer Science Part C: Paper C3.8**

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# ANALYTIC NUMBER THEORY

## Trinity Term 2015

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**TUESDAY, 9 JUNE 2015, 2.30pm to 4.00pm**

*You may submit answers to as many questions as you wish but only the best two will count for the total mark.*

*You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.*

**Do not turn this page until you are told that you may do so**

1. (a) [6 marks] Show from first principles that

$$\sum_{d|n} \phi(d) = n.$$

(b) [8 marks] What does it mean to say that an arithmetic function is *multiplicative*?
Define the *Dirichlet convolution* of two arithmetic functions.
Show that the function $\phi(n)$ is multiplicative, and that

$$\phi(n) = n \prod_{p|n} (1 - p^{-1})$$

where $p$ runs over primes. [*You may use standard results about multiplicative functions and Dirichlet convolutions, provided they are formally stated.*]

(c) [7 marks] Show that

$$\prod_{p \leqslant k} (1 + p^{-1} + p^{-2} + \ldots) \geqslant \sum_{m=1}^{k} m^{-1} \geqslant \log k$$

and deduce that if

$$n_k = \prod_{p \leqslant k} p$$

then

$$\phi(n_k) \leqslant \frac{n_k}{\log k}.$$

(d) [4 marks] Show that $n_k \leqslant k^k$ and hence that

$$\liminf_{n \to \infty} \frac{\phi(n) \log \log n}{n} \leqslant 1.$$

1. (a) [6 marks] Show from first principles that

$$\sum_{d|n} \phi(d) = n.$$

$\phi(n) = \#\{1 \leq k \leq n : (k, n) = 1\}$.

Partition $\{1, \ldots, n\}$ according to the size of gcd with $n$.

$$n = \sum_{d|n} \#\{j \leq n : \gcd(j, n) = d\} = \sum_{d|n} \phi\left(\frac{n}{d}\right).$$

by writing $j = j'd$ with $(j', \frac{n}{d}) = 1$.

(b) [8 marks] What does it mean to say that an arithmetic function is *multiplicative*?

Define the *Dirichlet convolution* of two arithmetic functions.

Show that the function $\phi(n)$ is multiplicative, and that

$$\phi(n) = n \prod_{p|n} (1 - p^{-1})$$

where $p$ runs over primes. [*You may use standard results about multiplicative functions and Dirichlet convolutions, provided they are formally stated.*]

Seen $\phi * 1 = \iota$ (identity function)

Fact: $1$ multiplicative

$\iota$ " "

$\mu$ " "

$$(1*\mu)(n) = \begin{cases} 1, & n=1 \\ 0, & n>1. \end{cases}$$

$$(a*b)*c = a*(b*c).$$

$$\therefore \phi = \phi * (1*\mu) = (\phi * 1) * \mu = \iota * \mu$$

FACT: convolution of 2 mult. functions is mult.

$\therefore \phi$ mult.

$\phi$ mult $\Rightarrow$ suff. to show $\phi(p^e) = p^e(1-p^{-1})$ for $e \geq 1$.

but $\#\{n \leq p^e : (n, p^e) = 1\} = \#\{n \leq p^e : (n,p) = 1\}$

$= p^e - p^{e-1}$.

(c) [7 marks] Show that

$$\prod_{p \leqslant k}(1 + p^{-1} + p^{-2} + \ldots) \geqslant \sum_{m=1}^{k} m^{-1} \geqslant \log k$$

and deduce that if

$$n_k = \prod_{p \leqslant k} p$$

then

$$\phi(n_k) \leqslant \frac{n_k}{\log k}.$$

Expanding out:

$$\prod_{p \leqslant k}(1 + p^{-1} + p^{-2} + \ldots) = \sum_{p \mid m \Rightarrow p \leqslant k} \frac{1}{m} \geqslant \sum_{m \leqslant k} \frac{1}{m}$$

$$\geqslant \int_1^{k+1} \frac{dt}{t}$$

$$\geqslant \log k.$$

Observe: $\dfrac{n_k}{\phi(n_k)} = \prod_{p \leqslant k}\left(\dfrac{p}{p-1}\right) = \prod_{p \leqslant k}\left(1 + \dfrac{1}{p} + \dfrac{1}{p^2} + \ldots\right) \geqslant \log k.$

$$\therefore \quad \phi(n_k) \leqslant \frac{n_k}{\log k}.$$

(d) [4 marks] Show that $n_k \leqslant k^k$ and hence that

$$\liminf_{n\to\infty} \frac{\phi(n)\log\log n}{n} \leqslant 1.$$

$$n_k = \prod_{p \leq k} p \leqslant k^{\pi(k)} \leqslant k^k .$$

$$\frac{\phi(n_k)\log\log n_k}{n_k} \leqslant \frac{\log\log n_k}{\log k} \leqslant \frac{\log k + \log\log k}{\log k}$$

$$= 1 + o_{k\to\infty}(1) .$$

**SECOND PUBLIC EXAMINATION**

**Honour School of Mathematics Part C: Paper C9.2a**

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# ANALYTIC NUMBER THEORY

## Trinity Term 2014

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**TUESDAY, 3 JUNE 2014, 2.30pm to 4.00pm**

*You may submit answers to as many questions as you wish but only the best two will count for the total mark.*

*You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.*

**Do not turn this page until you are told that you may do so**

1. (a) [2 marks] Let $f(n)$ and $g(n)$ be arithmetic functions. Define the *Dirichlet convolution* $(f * g)(n)$.

(b) [9 marks] Suppose that the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} f(n)n^{-s} \quad \text{and} \quad G(s) = \sum_{n=1}^{\infty} g(n)n^{-s}$$

are absolutely convergent for $\Re(s) > 1$. State and prove a theorem relating the Dirichlet series for $(f * g)(n)$ to $F(s)$ and $G(s)$. Use the theorem to show that

$$\sum_{n=1}^{\infty} d(n)n^{-s} = \zeta(s)^2 \quad (\Re(s) > 1).$$

(c) [4 marks] What does it mean to say that an arithmetic function is *multiplicative*? Assuming that $d(n)$ is a multiplicative function, show that

$$d(a)d(b) \geqslant d(ab)$$

for all positive integers $a$ and $b$.

(d) [3 marks] Deduce that $(d * d)(n) \geqslant d^2(n)$ for all natural numbers $n$.

(e) [7 marks] By using Dirichlet series with argument $s_0 = 1 + (\log x)^{-1}$ show that

$$\sum_{n \leqslant x} d(n)^2 n^{-1} \leqslant \mathrm{e} \sum_{n=1}^{\infty} d(n)^2 n^{-s_0} \leqslant \mathrm{e}(1 + \log x)^4$$

for any $x > 1$.

[*You may use without proof the fact that* $\zeta(\sigma) \leqslant 1 + (\sigma - 1)^{-1}$ *for* $\sigma > 1$.]

1. (a) [2 marks] Let $f(n)$ and $g(n)$ be arithmetic functions. Define the *Dirichlet convolution* $(f * g)(n)$.

   (b) [9 marks] Suppose that the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} f(n)n^{-s} \quad \text{and} \quad G(s) = \sum_{n=1}^{\infty} g(n)n^{-s}$$

are absolutely convergent for $\Re(s) > 1$. State and prove a theorem relating the Dirichlet series for $(f * g)(n)$ to $F(s)$ and $G(s)$. Use the theorem to show that

$$\sum_{n=1}^{\infty} d(n)n^{-s} = \zeta(s)^2 \quad (\Re(s) > 1).$$

- Bookwork !

(c) [4 marks] What does it mean to say that an arithmetic function is *multiplicative*? Assuming that $d(n)$ is a multiplicative function, show that

$$d(a)d(b) \geqslant d(ab)$$

for all positive integers $a$ and $b$.

(d) [3 marks] Deduce that $(d * d)(n) \geqslant d^2(n)$ for all natural numbers $n$.

c) $d(n)$ multiplicative $\Rightarrow$ if $a = p_1^{e_1} \cdots p_k^{e_k}$

$b = p_1^{f_1} \cdots p_k^{f_k}$ $(e_i, f_i \geq 0$ possibly $= 0)$

then $ab = p_1^{e_1+f_1} \cdots p_k^{e_k+f_k}$

$$\therefore \quad \frac{d(a)d(b)}{d(ab)} = \prod_{i=1}^{j} \frac{d(p_i^{e_i})d(p_i^{f_i})}{d(p_i^{e_i+f_i})}$$

$\therefore$ sufficient to show $\underbrace{d(p_i^{e_i})d(p_i^{f_i})}_{=(e_i+1)(f_i+1)} \geq \underbrace{d(p_i^{e_i+f_i})}_{=e_i+f_i+1}$

$\therefore$ this obvious as $e_i, f_i \geq 0$.

d) $$(d*d)(n) = \sum_{ab=n} d(a)d(b) \geq d(n) \sum_{ab=n} 1 = d(n)^2.$$

$$\frac{e}{n^{\delta}} = \frac{e}{n^{1/\log x}} \times \frac{1}{n} \quad \text{but } n \leq x \text{ so } n^{\frac{1}{\log x}} \leq x^{1/\log x} = e.$$

$$\therefore \frac{e}{n^{\delta}} \geq \frac{1}{n}.$$

(e) [7 marks] By using Dirichlet series with argument $s_0 = 1 + (\log x)^{-1}$ show that

$$\sum_{n \leqslant x} d(n)^2 n^{-1} \leqslant \mathrm{e} \sum_{n=1}^{\infty} d(n)^2 n^{-s_0} \leqslant \mathrm{e}(1 + \log x)^4$$

for any $x > 1$.

[*You may use without proof the fact that* $\zeta(\sigma) \leqslant 1 + (\sigma - 1)^{-1}$ *for* $\sigma > 1$.]

$$\sum_{n=1}^{\infty} \frac{d(n)^2}{n} \leqslant e \sum_{n=1}^{\infty} \frac{d(n)^2}{n^{s_0}} \qquad \left(\text{as } \frac{1}{n} \leqslant \frac{e}{n^{s_0}}\right)$$

$$\leqslant e \sum_{n=1}^{\infty} \frac{(d*d)(n)}{n^{s_0}} \qquad \left(\text{as } d(n)^2 \leqslant (d*d)(n)\right)$$

$$= e \left( \sum_{n=1}^{\infty} \frac{d(n)}{n^{s_0}} \right)^2 \qquad (\text{by Euler product})$$

$$= e \left( \sum_{n=1}^{\infty} \frac{(1*1)(n)}{n^{s_0}} \right)^2 \qquad \left(\text{as } d(n) = (1*1)(n)\right)$$

$$= e \left( \sum_{n=1}^{\infty} \frac{1}{n^{s_0}} \right)^4 \qquad (\text{by Euler product})$$

$$\leqslant e (1 + \log x)^4 \qquad \left(\text{since } \zeta(\sigma) \leqslant 1 + (\sigma-1)^{-1}\right)$$