Analytic Number Theory Revision Class 2

1 Summary

- Reminder: please email questions **4 days** prior to the classes.
- Next class: **Thu 21/05/20 15:00 - 16:00**, queries (maynard@maths.ox.ac.uk) **by the end of Sunday 17 May**.
- Final class: **Wed 27/05/20 10:00-11:00**, questions (wangr@maths.ox.ac.uk) by **the end of Saturday 24 May**
- Today: 2018 Q1, 2016 Q3, (time didn't permit) 2016 Q2.
- Questions related to this note: email wangr@maths.ox.ac.uk.

2 2018 Paper, Q1

(a) [4] Define the Riemann ζ-function $\zeta(s)$ for $\Re s > 1$, and show that it extends to a meromorphic function on $\Re s > 0$, holomorphic except for a simple pole at $s = 1$.
(b) [3] Show that

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s\int_N^\infty \{x\}x^{-1-s}dx$$

for any integer $N > 0$, and for all $\Re s > 0$.
(c) [3] Show that $|\zeta(\sigma + it)| = O(\log t)$, uniformly for $1 \leq \sigma \leq 2$ and for $t \geq 10$.
(d) [5] Show that $|\zeta'(\sigma + it)| = O(\log^2 t)$, uniformly for $1 \leq \sigma \leq 2$ and for $t \geq 10$.
(e) [10] Show that $|\zeta(1 + it)| \gg 1/\log^7 t$, uniformly for $t \geq 10$. [*You may use any part of the proof that ζ has no zero on $\Re s = 1$ provided you state it correctly.*]

(a) **Proof.**

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

for $\Re s > 1$. By partial summation (see Lemma 4.2 and Lemma 2.2 course notes),

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = s\int_1^\infty \frac{x - \{x\}}{x^{s+1}}dx = s\int_1^\infty \frac{1}{x^s}dx - s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx$$
$$= \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx.$$

The integral on the RHS converges absolutely.

(b) [3] Show that

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s\int_N^\infty \{x\}x^{-1-s}dx$$

for any integer $N > 0$, and for all $\Re s > 0$.

Proof. Again use partial summation:

$$\zeta(s) = \sum_{n\leq N} n^{-s} + s\int_N^\infty \frac{x-\{x\}}{x^{s+1}}dt = \sum_{n\leq N} n^{-s} + s\int_N^\infty \frac{1}{x^s}dx - s\int_N^\infty \frac{\{x\}}{x^{s+1}}dx$$

$$= \sum_{n\leq N} n^{-s} + \frac{N^{1-s}}{s-1} - s\int_N^\infty \frac{\{x\}}{x^{s+1}}dx.$$

(c) [3] Show that $|\zeta(\sigma+it)| = O(\log t)$, uniformly for $1 \le \sigma \le 2$ and for $t \ge 10$.

Proof. Bound each term of the identity

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s\int_N^\infty \{x\}x^{-1-s}dx$$

as follows:

$$|\sum_{n=1}^{N} \frac{1}{n^s}| = O(\log N),$$

$$|\frac{N^{1-s}}{s-1}| = O(1),$$

$$|s\int_N^\infty \{x\}x^{-1-s}dx| \ll t\int_N^\infty x^{-2}dx \ll \frac{t}{N}.$$

Take $N = t$ to obtain the result.

(d) [5] Show that $|\zeta'(\sigma+it)| = O(\log^2 t)$, uniformly for $1 \le \sigma \le 2$ and for $t \ge 10$.

Proof. Differentiate the result from (b) with respect to s.

$$\zeta'(s) \quad = \quad -\sum_{n\le N} n^{-s}\log n \; - \; \frac{N^{1-s}}{(s-1)^2} \; - \; \frac{N^{1-s}\log N}{s-1} \; - \; \int_N^\infty \frac{\{x\}}{x^{s+1}}dx \; - \; s\int_N^\infty \frac{\{x\}\log x}{x^{s+1}}dx$$

The first four terms are bounded by $O(\log^2 N), O(1), O(\log N), O(1/N)$ respectively. The final term:

$$s|\int_N^\infty \frac{\{x\}\log x}{x^{s+1}}dx| \ll t\int_N^\infty x^{-3/2}dx \ll tN^{-1/2}.$$

Take $N = \lfloor t^2 \rfloor$ so the first term dominates. It follows that

$$|\zeta'(\sigma+it)| = O(\log^2 t) \tag{18.1.1}$$

Notes: uniformly – there exists absolute constant C such that $\zeta(\sigma+it) \le C\log t$ for all t.

(e) [10] Show that $|\zeta(1+it)| \gg 1/\log^7 t$, uniformly for $t \geq 10$. [*You may use any part of the proof that* ζ *has no zero on* $\Re s = 1$ *provided you state it correctly.*]

Proof. Perhaps the easiest thing to do is to use the estimate (Lemma 11.1). For $\sigma > 1$,

$$4\Re(\frac{\zeta'}{\zeta}(\sigma+it)) + \Re(\frac{\zeta'}{\zeta}(\sigma+2it)) + 3\frac{\zeta'}{\zeta}(\sigma) \leq 0.$$

Integrate both sides in σ (note $\zeta(\sigma+it) \to 1$ as $\sigma \to \infty$ for any t)

$$4\Re(\log(\zeta(\sigma+it))) + \Re(\log(\zeta(\sigma+2it))) + 3\log\zeta(\sigma) \geq 0;$$

take exponents

$$|\zeta(\sigma+it)|^4|\zeta(\sigma+2it)|(\zeta(\sigma))^3 \geq 1.$$

Take $\sigma = 1+\epsilon$ where ϵ is a parameter to be specified later. The identity above yields

$$|\zeta(1+\epsilon+it)| \geq |\zeta(1+\epsilon+2it)|^{-1/4}(\zeta(1+\epsilon))^{-3/4}.$$

Use estimates from (c) and $\zeta(1+\epsilon) \sim 1/\epsilon$ (see part (a)), the RHS is at least

$$|\zeta(1+\epsilon+it)| \geq (\log t)^{-1/4}\epsilon^{3/4}. \qquad (18.1.2)$$

To deduce the result for $1+it$, use part (d) and the mean value theorem to deduce

$$|\zeta(1+it)| \geq c(\log t)^{-1/4}\epsilon^{3/4} - C\epsilon\log^2 t,$$

where c, C are implicit constants from (18.1.2) and the upper bound on the derivative (18.1.1). Take $\epsilon = c'\log^{-9} t$ for some sufficiently small absolute constant c' to obtain the result.

3 2016 Paper, Q3

(a) [7 marks] Give, without proof, the value of the integral

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{y^s}{s(s+1)}\mathrm{d}s,$$

where $c > 1$ and $y > 0$.
Show that if $c > 1$ and $x > 0$ then

$$\sum_{n\le x}\Lambda(n)(x-n) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left\{-\frac{\zeta'(s)}{\zeta(s)}\right\}\frac{x^{s+1}}{s(s+1)}\mathrm{d}s.$$

(b) [12 marks] Assuming that $1/\zeta(s) = \sum_{n=1}^{\infty}\mu(n)n^{-s}$ is absolutely convergent for $\Re(s) > 1$, state a standard theorem which allows you to deduce the Dirichlet series (which you should give) for

$$F(s) := \frac{d}{ds}\left(\frac{1}{\zeta(s)}\right).$$

Quoting appropriate theorems from the course, give an upper bound for $F(s)$, independent of $\sigma = \Re(s)$, valid in the region $1 < \sigma \le 2$, $|t| \ge 2$.
For the rest of the question you may assume without proof that $F(s) = O(1)$ for $1 < \sigma \le 2$ and $|t| \le 2$.
Given the formula

$$\sum_{n\le x}\mu(n)(\log n)(x-n) = \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\frac{x^{s+1}}{s(s+1)}\mathrm{d}s$$

(valid for any $c > 1$ and $x > 0$), which you may assume without proof, show that

$$\sum_{n\le x}\mu(n)(\log n)(x-n) = O(x^2).$$

(a) [7 marks] Give, without proof, the value of the integral

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{y^s}{s(s+1)}\mathrm{d}s,$$

where $c>1$ and $y>0$.
Show that if $c>1$ and $x>0$ then

$$\sum_{n\le x}\Lambda(n)(x-n)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left\{-\frac{\zeta'(s)}{\zeta(s)}\right\}\frac{x^{s+1}}{s(s+1)}\mathrm{d}s.$$

Proof. We have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{y^s}{s(s+1)}\mathrm{d}s=\begin{cases}0 & \text{if } y\le 1;\\ 1-y^{-1} & \text{otherwise.}\end{cases}$$

This can be deduced using Perron's formula (Chap. 8 notes; you don't have to clarify this for this exam question).
Note that $-\zeta'(s)/\zeta(s)=\sum_{n=1}^{\infty}\Lambda(n)n^{-s}$ for $\Re s>1$. By Fubini's theorem one may integrate termwisely if one can show that

$$\sum_{n=1}^{\infty}\Lambda(n)\int_{c-i\infty}^{c+i\infty}\left|\frac{x^{s+1}n^{-s}}{s(s+1)}\right|\mathrm{d}s$$

is finite. Since $c>1$, this integral is bounded by

$$\sum_{n=1}^{\infty}\Lambda(n)x^{c+1}n^{-c}\left(\int_{-\infty}^{\infty}\left|\frac{1}{t^2+1}\right|\mathrm{d}t\right)\ll\sum_{n=1}^{\infty}\Lambda(n)x^{c+1}n^{-c}<\infty.$$

Thus the right hand is

$$\begin{aligned}&\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left\{-\frac{\zeta'(s)}{\zeta(s)}\right\}\frac{x^{s+1}}{s(s+1)}\mathrm{d}s\\ =&\sum_{n=1}^{\infty}\Lambda(n)\int_{c-i\infty}^{c+i\infty}\frac{x}{2\pi i}\frac{(x/n)^s}{s(s+1)}\mathrm{d}s\\ =&\sum_{n<x}\Lambda(n)(x-n). \qquad \text{(use the earlier part of this question)}\end{aligned}$$

Notes: for $c=1+\epsilon$, we have $\Lambda(n)n^{-c}\le(\log n)n^{-1-\epsilon}\ll n^{-1-\epsilon/2}$.

(b) [12 marks] Assuming that $1/\zeta(s) = \sum_{n=1}^{\infty} \mu(n)n^{-s}$ is absolutely convergent for $\Re(s) > 1$, state a standard theorem which allows you to deduce the Dirichlet series (which you should give) for

$$F(s) := \frac{d}{ds}\left(\frac{1}{\zeta(s)}\right).$$

Proof. Let $s = \sigma + it$. If $f(s) = a_n n^{-s}$ is absolutely convergent for $\sigma > c$, then $f(s)$ is holomorphic in this region and $f'(s) = -a_n(\log n)n^{-s}$. Use this to deduce that $F(s) = -\sum_{n=1}^{\infty} \mu(n)(\log n)n^{-s}$. Alternatively, use $1/\zeta(s) = \sum_{n=1}^{\infty} \mu(n)n^{-s}$, $-\zeta'(s)/\zeta(s) = \sum_{n=1}^{\infty} \Lambda(n)n^{-s}$ and Lemma 5.3 (Dirichlet series of convolution is product of Dirichlet series) to show $F(s) = -\sum_{n=1}^{\infty}(\mu \star \Lambda)n^{-s}$. Use PS2Q5 to compute the Dirichlet convolution.

Quoting appropriate theorems from the course, give an upper bound for $F(s)$, independent of $\sigma = \Re(s)$, valid in the region $1 < \sigma \leq 2$, $|t| \geq 2$.

Proof. The estimates needed are the ones proved in Q1, 2018 (you can use the same proof to obtain the estimates for $|t| \geq 2$, with different implicit constant):
- $|\zeta'(\sigma + it)| = O(\log^2 t)$, uniformly for $1 \leq \sigma \leq 2$ and for $|t| \geq 2$;
- $|\zeta(\sigma + it)| \gg 1/\log^7 t$, uniformly for $|t| \geq 2$.
Hence $F(s) = -\zeta'(s)/\zeta^2(s) = O(\log^{16}|t|)$ for $1 < \sigma \leq 2$, $|t| \geq 2$.

For the rest of the question you may assume without proof that $F(s) = O(1)$ for $1 < \sigma \le 2$ and $|t| \le 2$.
Given the formula

$$\sum_{n \le x} \mu(n)(\log n)(x-n) = \frac{-1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \frac{x^{s+1}}{s(s+1)} \mathrm{d}s$$

(valid for any $c > 1$ and $x > 0$), which you may assume without proof, show that

$$\sum_{n \le x} \mu(n)(\log n)(x-n) = O(x^2).$$

Proof. Take $c = 1 + 1/\log x$. Decompose the integral into two parts:

$$\int_{c-i\infty}^{c+i\infty} F(s) \frac{x^{s+1}}{s(s+1)} \mathrm{d}s = \Big(\int_{|t| \ge 2} + \int_{|t| \le 2} \Big) F(c+it) \frac{x^{c+it+1}}{(c+it)(c+it+1)} \mathrm{d}t.$$

Bound the first part use $F(s) = O(\log^{16} |t|)$ and bound the logarithm by any small power of t:

$$\Big| \int_{|t| \ge 2} F(c+it) \frac{x^{c+it+1}}{(c+it)(c+it+1)} \mathrm{d}t \Big| \ll \int_{|t| \ge 2} \frac{\log^{16}(t) x^{c+1}}{t^2} dt = O(x^2).$$

For the second part, use $F(s) = O(1)$ and $|s(s+1)| \gg 1$ in this region to deduce that

$$\Big| \int_{|t| \le 2} F(c+it) \frac{x^{c+it+1}}{(c+it)(c+it+1)} \mathrm{d}t \Big| \ll \int_{|t| \le 2} x^{c+1} dt \ll x^{c+1} \le e x^2.$$

Notes: c is chosen to obtain the $O(x^2)$ at the end.

4 2016 Paper, Q2

(a) [12 marks] For any complex s with $\Re(s) > 0$ and any $n \in \mathbb{N}$, let

$$f_n(s) = \int_n^{n+1} \frac{x-n}{x^{s+1}} \mathrm{d}x$$

and

$$\zeta^*(s) = \frac{s}{s-1} - s \sum_{n=1}^{\infty} f_n(s).$$

Show that $\zeta^*(s)$ is defined and holomorphic for $\Re(s) > 0$, except for a simple pole at $s = 1$.
Show by induction on M that

$$\zeta^*(s) = \frac{M^{1-s}}{s-1} + \sum_{n=1}^{M} n^{-s} - s \sum_{n=M}^{\infty} f_n(s)$$

for any $M \in \mathbb{N}$.
Deduce by taking $M \to \infty$ that $\zeta^*(s) = \zeta(s)$ for $\Re(s) > 1$.
(b) [4 marks] For the rest of this question we use the standard convention by which we merely write $\zeta(s)$ in place of $\zeta^*(s)$.
Show that

$$\sum_{n=1}^{M} n^{-1/2} = 2\sqrt{M} + \zeta(1/2) + O(M^{-1/2})$$

for any $M \in \mathbb{N}$.
(c) [5 marks] By choosing a suitable value for M show that

$$\zeta(1/2 + it) = O(\sqrt{t})$$

for $t \geq 2$.
(d) [4 marks] Show that if s is real, with $0 < s < 1$, then $\zeta(s) \neq 0$.
Notes: due to time limit, we shall only provide a sketch of the proof of this question.

(a) [12 marks] For any complex s with $\Re(s) > 0$ and any $n \in \mathbb{N}$, let

$$f_n(s) = \int_n^{n+1} \frac{x-n}{x^{s+1}} \mathrm{d}x$$

and

$$\zeta^*(s) = \frac{s}{s-1} - s \sum_{n=1}^{\infty} f_n(s).$$

Show that $\zeta^*(s)$ is defined and holomorphic for $\Re(s) > 0$, except for a simple pole at $s = 1$.
Show by induction on M that

$$\zeta^*(s) = \frac{M^{1-s}}{s-1} + \sum_{n=1}^{M} n^{-s} - s \sum_{n=M}^{\infty} f_n(s)$$

for any $M \in \mathbb{N}$.
Deduce by taking $M \to \infty$ that $\zeta^*(s) = \zeta(s)$ for $\Re(s) > 1$.

Sketch of Proof. The function f_n is holomorphic on $\mathbb{C}$. By Weierstrass' Lemma, $\sum_{n=1}^{\infty} f_n(s)$ is convergent on $\Re s \geq c$ provided the sum is uniformly convergent. Check this for all $c \geq 0$ and conclude that the function is holomorphic on $\Re(s) > 0$.
Induction: The base case

$$\zeta^*(s) = \frac{1}{s-1} + 1 - s \sum_{n=1}^{\infty} f_n(s).$$

For the induction step, one needs

$$\frac{(M+1)^{1-s}}{s-1} + (M+1)^{-s} = (M)^{1-s}s - 1 - sf_M(s).$$

This follows from direct computation of f_M.
Let $\Re s > 1$. By taking $M \to \infty$,

$$\frac{M^{1-s}}{s-1} \to 0;$$

$$\sum_{n=1}^{M} n^{-s} \to \zeta(s)$$

$$s \sum_{n=M}^{\infty} f_n(s) \to 0 \text{ since } \sum_{n=1}^{\infty} f_n(s) \text{ converges.}$$

(b) [4 marks] For the rest of this question we use the standard convention by which we merely write $\zeta(s)$ in place of $\zeta^*(s)$.
Show that

$$\sum_{n=1}^{M} n^{-1/2} = 2\sqrt{M} + \zeta(1/2) + O(M^{-1/2})$$

for any $M \in \mathbb{N}$.
Sketch of Proof. By (a)

$$\zeta(1/2) = -\frac{M^{1/2}}{1/2} + \sum_{n=1}^{M} n^{-1/2} - \sum_{n=M}^{\infty} \frac{f_n(1/2)}{2},$$

but

$$f_n(1/2) = \int_n^{n+1} \frac{x-n}{x^{s+1}} dx \ll \int_n^{n+1} x^{-3/2} dx$$

and so

$$\sum_{n=M}^{\infty} \frac{f_n(1/2)}{2} \ll \int_M^{\infty} x^{-3/2} dx \ll M^{-1/2}.$$

(c) [5 marks] By choosing a suitable value for M show that

$$\zeta(1/2 + it) = O(\sqrt{t})$$

for $t \geq 2$.
Sketch of Proof. Take M to be the largest integer less than t and perform similar estimates as before.
(d) [4 marks] Show that if s is real, with $0 < s < 1$, then $\zeta(s) \neq 0$.
Sketch of Proof. For $0 < s < 1$ we have $s/(s-1)) < 0$ and $f_n(s) > 0$, so $\zeta(s) < 0$ by (a).