Analytic Number Theory

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CHAPTER 1

Asymptotic Estimates

We will repeatedly encounter interesting number-theoretic objects which are complicated, such as the counting function of the primes. To understand these complicated functions, we want to approximate them by much simpler functions, such as a continuous function with no number-theoretic properties. To do this we need to control the error in such approximations, and the following notation is very useful to keep us focused on what is going on.

Definition (Big Oh notation). *We write '$O(h(x))$' to denote a function $g(x)$ which satisfies*

$$|g(x)| \leq C \cdot h(x)$$

for some constant $C > 0$ and all x under consideration.

Since the function g and the constant C are unspecified, multiple uses of $O(\cdot)$ can specify different functions. Moreover, this can lead to some initally confusing issues when used with the $=$ sign, since $f(x) = O(h(x))$ and $g(x) = O(h(x))$ does not imply that $f(x) = g(x)$. Moreover, we will use $O(h(x))$ inside various expressions, so given functions f, g, h, when we write '$f(x) = g(x) + O(h(x))$ for $x \in \mathcal{S}$' we mean there exists a constant $C > 0$ (which depends only on $f, g, h, \mathcal{S}$) such that

$$|f(x) - g(x)| \leq C \cdot h(x)$$

for all $x \in \mathcal{S}$. If the set $\mathcal{S}$ is clear from the context (as is normally the case), we just write '$f(x) = g(x) + O(h(x))$'. We sometimes call $g(x)$ the 'main term' and $h(x)$ the 'error term' in an approximation to f.

Example 1.1.

- $x = O(x^2)$ *for* $x \geq 1$. *(Since $x \leq x^2$ for $x \geq 1$.)*
- $x^2 = O(x)$ *for* $0 \leq x \leq 10$. *(Since $x^2 \leq 10x$ for $0 \leq x \leq 10$.)*
- *It is not the case that $x^2 = O(x)$ for $x \geq 1$ (since as $x \to \infty$, $x^2/x \to \infty$.)*
- $(x+1)^2 = x^2 + O(x)$ *for* $x \geq 1$ *(since $|(x+1)^2 - x^2| \leq 3x$ for $x \geq 1$.)*
- $\lfloor x \rfloor = \sup\{n \in \mathbb{Z} : n \leq x\} = x + O(1)$ *for* $x \in \mathbb{R}$. *(Since $x - 1 \leq \lfloor x \rfloor \leq x$, so $|\lfloor x \rfloor - x| \leq 1$.)*

- $\sqrt{x+1} = \sqrt{x} + \frac{1}{2\sqrt{x}} - \frac{1}{8x^{3/2}} + O\Big(\frac{1}{x^{5/2}}\Big)$ *for* $x \geq 1$. *(Since for* $f(x) = \sqrt{x}$, $f(x+1) = f(x) + f'(x) + f''(x)/2 + f'''(y)/6$ *for some* $y \in [x, x+1]$ *by Taylor's Theorem, and* $f'''(y) = 3/(8y^{5/2}) \leq 6/(8x^{5/2})$ *for* $x \geq 1$.*)*

Lemma 1.2 (Properties of Big Oh notation).

(1) Non-negativity of error term:
If $f(x) = O(g(x))$ *then* $g(x) \geq 0$.
(2) Transitivity:
If $f(x) = O(g(x))$ *and* $g(x) = O(h(x))$ *then* $f(x) = O(h(x))$.
(3) Additivity:
If $f_1(x) = g_1(x) + O(h_1(x))$ *and* $f_2(x) = g_2(x) + O(h_2(x))$ *then*
$f_1(x) + f_2(x) = g_1(x) + g_2(x) + O(h_1(x) + h_2(x))$.

Proof. These follow immediately from the definition. □

Definition (Further asymptotic notation).

- *Little Oh notation:*
Given $h(x) > 0$, *when considering a limit* $x \to a$ *we write '*$o(h(x))$*' to denote a function* $g(x)$ *which satisfies*
$$\lim_{x\to a} \frac{g(x)}{h(x)} \to 0.$$
If we don't explicitly mention the limit point a *then it is assumed* $a = \infty$.
- *Vinogradov notation:*
We have the binary relation $f(x) \ll g(x)$ *if* $f(x) = O(g(x))$.
- *For two positive functions* f, g, *we write* $f(x) \sim g(x)$ *as* $x \to a$ *if*
$$\lim_{x\to a} \frac{f(x)}{g(x)} = 1.$$
If we just write $f(x) \sim g(x)$ *then it is assumed* $a = \infty$.
- *We write* $f(x) \asymp g(x)$ *for* $x \in \mathcal{S}$ *if* $f(x) = O(g(x))$ *for* $x \in \mathcal{S}$ *and* $g(x) = O(f(x))$ *for* $x \in \mathcal{S}$.

Although the Vinogradov notation overlaps with Big Oh notation, the Big Oh notation should be thought of as a placeholder for some unspecified function, whereas the $\ll$ is an inequality which can exploit the transitivity of $O(\cdot)$, so we might write things like $f(x) \ll g(x) \ll h(x)$.

CHAPTER 2

Partial Summation

LEMMA 2.1 (Partial Summation). *Let $a_n \in \mathbb{C}$ be a complex sequence, and $f : \mathbb{R} \to \mathbb{R}$ continuously differentiable on the interval $[x, y]$. Let*

$$A(t) := \sum_{n \le t} a_n.$$

Then

$$\sum_{x<n\le y} a_n f(n) = A(y)f(y) - A(x)f(x) - \int_x^y A(t)f'(t)dt.$$

PROOF. We let $n_1 = \lfloor x \rfloor + 1$, $n_2 = \lfloor y \rfloor$ so the sum is over $n_1 \le n \le n_2$. (Here $\lfloor z \rfloor$ is the largest integer less than or equal to z.) We note that $a_n = A(n) - A(n-1)$. Therefore, rearranging the sums, we see that

$$\begin{aligned}\sum_{x<n\le y} a_n f(n) &= \sum_{n_1\le n\le n_2} f(n)\Big(A(n) - A(n-1)\Big) \\ &= \sum_{n_1\le n\le n_2} f(n)A(n) - \sum_{n_1-1\le n\le n_2-1} f(n+1)A(n) \\ &= \sum_{n_1\le n\le n_2-1} A(n)\Big(f(n) - f(n+1)\Big) + A(n_2)f(n_2) - A(n_1-1)f(n_1).\end{aligned}$$

Since f is differentiable, $\int_n^{n+1} f'(t)dt = f(n+1) - f(n)$. Therefore

$$\sum_{n_1\le n\le n_2-1} A(n)\Big(f(n) - f(n+1)\Big) = -\sum_{n_1\le n\le n_2-1} A(n)\int_n^{n+1} f'(t)dt.$$

Since $A(t)$ only changes at integers, $A(t) = A(n)$ for $t \in [n, n+1)$. Therefore

$$\sum_{n_1\le n\le n_2-1} A(n)\int_n^{n+1} f'(t)dt = \sum_{n_1\le n\le n_2-1}\int_n^{n+1} A(t)f'(t)dt = \int_{n_1}^{n_2} A(t)f'(t)dt.$$

This gives

$$\sum_{x<n\le y} a_n f(n) = A(n_2)f(n_2) - A(n_1-1)f(n_1) - \int_{n_1}^{n_2} A(t)f'(t)dt.$$

This is essentially the result; to finish off we just need to observe that

$$A(y)f(y) - \int_{n_2}^{y} A(t)f'(t) = A(n_2)f(n_2),$$

and

$$-A(x)f(x) - \int_x^{n_1} A(t)f'(t)dt = -A(n_1-1)f(n_1)$$

since $A(t) = A(n_2) = A(y)$ for $t \in [n_2, y]$ and $A(t) = A(x) = A(n_1 - 1)$ for $t \in [x, n_1)$. □

COROLLARY 2.2. *Let $a_n, f, A(t)$ be as in Lemma 2.1. If $A(t)f(t) \to 0$ as $t \to \infty$ then*

$$\sum_{n=1}^{\infty} a_n f(n) = -\int_1^{\infty} A(t)f'(t)dt$$

whenever both sides converge.

PROOF. Apply Lemma 2.1 with $x = 1-\epsilon$ and $y = 1/\epsilon$, and then let $\epsilon \to 0$. □

LEMMA 2.3. *Let $\pi(x) = \#\{p < x\}$ be the prime counting function, and $\theta(x) = \sum_{p<x} \log p$. Then we have*

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)dt}{t(\log t)^2}.$$

In particular, if $\theta(x) = x + o(x)$ then

$$\pi(x) = \frac{x}{\log x} + o\Big(\frac{x}{\log x}\Big),$$

and if $\theta(x) = x + O(x^{1/2}(\log x)^2)$ then

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O(x^{1/2}\log x).$$

PROOF. Let $a_n = \log n$ if n is prime, and 0 otherwise. Let $f(t) = 1/\log t$. Then by Lemma 2.1

$$\pi(y) = \sum_{n<y} a_n f(n) = \frac{\sum_{p<y}\log p}{\log y} + \int_2^y \Big(\sum_{p<t} \log p\Big) \frac{dt}{t(\log t)^2}.$$

This gives the first statement. If $\theta(t) = t + o(t)$ then this gives

$$\pi(y) = \frac{y + o(y)}{\log y} + O\Big(\int_2^y \frac{dt}{(\log t)^2}\Big) = \frac{y}{\log y} + o\Big(\frac{y}{\log y}\Big).$$

If $\theta(t) = t + O(t^{1/2}(\log t)^2)$ then this gives

$$\pi(y) = \frac{y + O(y^{1/2}(\log y)^2)}{\log y} + \int_2^y \frac{dt}{(\log t)^2} + O\Big(\int_2^y \frac{dt}{t^{1/2}}\Big)$$
$$= \frac{y}{\log y} + \int_2^y \frac{dt}{(\log t)^2} + O(y^{1/2}\log y).$$

To finish we see that integration by parts gives

$$\int_2^y \frac{dt}{\log t} = \frac{y}{\log y} - \frac{2}{\log 2} + \int_2^y \frac{dt}{(\log t)^2}.$$

□

CHAPTER 3

Arithmetic Functions

Definition (Multiplicative functions). *Let $f : \mathbb{Z}_{>0} \to \mathbb{C}$ be a function on the positive integers. We say that f is* multiplicative *if $f(nm) = f(n)f(m)$ for any coprime integers n, m. We say that f is* completely multiplicative *if $f(nm) = f(n)f(m)$ for all integers n, m.*

Definition (Dirichlet convolution). *Let $f,\ g : \mathbb{Z}_{>0} \to \mathbb{C}$. Then the Dirichlet convolution $f \star g$ is a function defined by*

$$(f \star g)(n) = \sum_{ab=n} f(a)g(b).$$

Lemma 3.1 (Basic properties of Dirichlet convolution). *Let $f, g, h : \mathbb{Z}_{>0} \to \mathbb{C}$. Then*

1. *Dirichlet convolution is commutative; $f \star g = g \star f$.*
2. *Dirichlet convolution is associative; $(f \star g) \star h = f \star (g \star h)$.*
3. *Dirichlet convolution preserves multiplicativity; If f and g are multiplicative then $f \star g$ is multiplicative.*

Proof. These follow from the definitions:

$$(f \star g)(n) = \sum_{n=ab} f(a)g(b) = \sum_{n=ab} f(b)g(a) = (g \star f)(n),$$
$$((f \star g) \star h)(n) = \sum_{n=ab} h(b) \sum_{a=cd} f(c)g(d) = \sum_{n=bcd} h(b)f(c)g(d) = (f \star (g \star h))(n),$$

If $\gcd(n_1, n_2) = 1$ then, letting $a_1 = \gcd(a, n_1)$ and $a_2 = a/a_1$ (and similarly for b)

$$\begin{aligned}(f \star g)(n_1 n_2) = \sum_{ab=n_1 n_2} f(a)g(b) &= \sum_{\substack{a_1 b_1 = n_1 \\ a_2 b_2 = n_2}} f(a_1 a_2) g(b_1 b_2) \\ &= \sum_{\substack{a_1 b_1 = n_1 \\ a_2 b_2 = n_2}} f(a_1)g(b_1)f(a_2)g(b_2) \\ &= (f \star g)(n_1) \cdot (f \star g)(n_2).\end{aligned}$$ ☐

Definition (Special arithmetic functions μ, Λ, τ). *We have the following definitions:*

- *The Möbius function $\mu(n)$ is $(-1)^k$ if n is a product of k distinct primes, and 0 if n has a repeated prime factor (and $\mu(1) = 1$).*
- *The Von Mangoldt function $\Lambda(n)$ is $\log p$ if $n = p^j$ for some prime p, and 0 if n has two or more distinct prime factors (and $\Lambda(1) = 0$).*
- *The Divisor function $\tau(n)$ is the number of different ways of writing $n = ab$ for two positive integers a, b.*

Lemma 3.2 (Mobius inversion). *If $f, g : \mathbb{Z}_{>0} \to \mathbb{C}$ then $f = g \star 1$ if and only if $g = f \star \mu$.*

Proof. Let $\delta(n) = 1$ if $n = 1$ and 0 otherwise. If $n = p_1^{e_1} \dots p_k^{e_k} > 1$ then

$$(\mu\star 1)(n) = \sum_{d|n} \mu(d) = \sum_{\substack{d_1,\dots d_k \\ d_i | p_i^{e_i}}} \mu(d_1)\dots\mu(d_k) = \prod_{i=1}^{k}\Big(\mu(1)+\dots+\mu(p_i^{e_i})\Big) = (1-1)^k = 0.$$

If $n = 1$ then $(\mu \star 1)(n) = 1$. Thus $\mu \star 1 = \delta$. Now if $g = f \star 1$ then

$$g \star \mu = (f \star 1) \star \mu = f \star (\mu \star 1) = f \star \delta = f.$$

Conversely, if $f = g \star \mu$ then

$$f \star 1 = (g \star \mu) \star 1 = g \star (\mu \star 1) = g \star \delta = g.$$ □

Lemma 3.3.

$$\Lambda(n) = (\mu \star \log)(n).$$

Proof. Let n have prime factorization $n = p_1^{e_1} \dots p_k^{e_k}$ for distinct primes $p_1, \dots, p_k$. Then

$$\log n = \sum_{i=1}^{k} e_i \log p_i = \sum_{i=1}^{k} \sum_{\substack{d = p_i^j > 1 \\ d|n}} \log p_i = \sum_{d|n} \Lambda(d) = (\Lambda \star 1)(n).$$

Now the result follows by Möbius inversion (Lemma 3.2). □

Lemma 3.4. *Let $\psi(x) = \sum_{n<x} \Lambda(n)$. Then we have for $x \geq 2$*

$$|\psi(x) - \theta(x)| \ll x^{1/2} \log x.$$

Proof. Recall than Λ is non-zero only on prime powers. We split the contributions to ψ according to the exponent:

$$\psi(x) = \sum_{1 \leq j \leq \log x / \log 2} \sum_{\substack{n<x \\ n=p^j}} \log p = \sum_{p<x} \log p + \sum_{p<x^{1/2}} \log p + \sum_{3 \leq j \leq 2\log x} \sum_{p \leq x^{1/j}} \log p.$$

The first term is exactly $\theta(x)$. The other terms are bounded by

$$\sum_{n<x^{1/2}} \log x + \sum_{3\leq j \leq 2 \log x} \sum_{n<x^{1/3}} \log x \ll x^{1/2} \log x + x^{1/3} (\log x)^2 \ll x^{1/2} \log x.$$ □

CHAPTER 4

Dirichlet Series

Lemma 4.1 (Region of absolute convergence). *Let $\delta > 0$ and $f : \mathbb{Z} \to \mathbb{C}$ satisfy $|f(n)| \leq n^{o(1)}$. Then the series*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

converges absolutely to an analytic function for $\Re(s) > 1$, and uniformly absolutely on $\Re(s) \geq 1 + \delta$.

Proof. Since $f(n) = n^{o(1)}$, there is an $N_0(\delta)$ such that $|f(n)| \leq n^{\delta/2}$ for $n \geq N_0$. If $\Re(s) \geq 1 + \delta$ then for $N_2 \geq N_1 \geq N_0(\delta)$ we have

$$\sum_{n=N_1}^{N_2} \frac{|f(n)|}{|n^s|} \leq \sum_{n=N_1}^{N_2} \frac{n^{\delta/2}}{n^{\Re(s)}} \leq \int_{N_1-1}^{N_2} \frac{ds}{t^{1+\delta/2}} \leq \frac{2}{\delta(N_1-1)^{\delta/2}}.$$

For fixed $\delta > 0$, this tends to 0 as $N_1 \to \infty$. Thus the series converges uniformly absolutely in the region $\Re(s) \geq 1 + \delta$. Since $\delta > 0$ was arbitrary, and the partial sums are clearly analytic, this gives the result. □

Definition (The Riemann Zeta function). *$\zeta(s)$ is defined for $\Re(s) > 1$ by*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Although the series $\sum_{n=1}^{\infty} n^{-s}$ no longer converges absolutely when $\Re(s) \leq 1$, we find that we can extend the definition of $\zeta(s)$ to a larger region.

Lemma 4.2 (Analytic Continuation of $\zeta(s)$). *The function $\zeta(s)$ has a meromorphic continuation to the region $\Re(s) > -2$. In this region we have that*

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \frac{s}{12} - s(s+1)(s+2) \int_1^{\infty} \frac{(\{t\} - 3\{t\}^2 + 2\{t\}^3)dt}{12t^{s+3}}$$

Here $\{t\} = t - \lfloor t \rfloor$ is the fractional part of t.

PROOF. We apply Lemma 2.2 with $a_n = 1$, $f(n) = n^{-s}$, $x = 1 - \epsilon$ and $y = \infty$. With this choice $A(t) = \lfloor t \rfloor = t - \{t\}$. This gives

$$\zeta(s) = \sum_{n=1}^{\infty} a_n f(n) = s \int_1^{\infty} \frac{(t - \{t\})dt}{t^{s+1}} = s \int_1^{\infty} \frac{1}{t^s} - s \int_1^{\infty} \frac{\{t\}dt}{t^{s+1}}$$
$$= \frac{s}{s-1} - s \int_1^{\infty} \frac{\{t\}dt}{t^{s+1}}.$$

The function $s/(s-1)$ is meromorphic in the entire complex plane, with a simple pole at $s = 1$. For $\Re(s) > 0$ the integral on the right hand side converges absolutely. Thus the right hand side defines a function on $\Re(s) > 0$ with a simple pole at $s = 1$ and analytic elsewhere, which coincides with $\sum_{n=1}^{\infty} n^{-s}$ for $\Re(s) \geq 1$. We can extend this further by integration by parts. For $\Re(s) > -1$ we have

$$\frac{s}{s-1} - s \int_1^{\infty} \frac{\{t\}dt}{t^{s+1}} = \frac{1}{s-1} + \frac{1}{2} - s \int_1^{\infty} \frac{(\{t\} - 1/2)dt}{t^{s+1}}$$
$$= \frac{1}{s-1} + \frac{1}{2} - s(s+1) \int_1^{\infty} \frac{g(t)dt}{t^{s+2}}$$

where $g(t) = \int_0^t (\{u\} - 1/2)du = (\{t\}^2 - \{t\})/2$ (note that $|g(t)| \leq 1/8$ for all t). Continuing once more gives

$$\frac{1}{s-1} + \frac{1}{2} + \frac{s}{12} - s(s+1)(s+2) \int_1^{\infty} \frac{(\{t\} - 3\{t\}^2 + 2\{t\}^3)dt}{12t^{s+3}}. \qquad \square$$

COROLLARY 4.3. *If* $0 < x < 1$ *then* $\zeta(x) < 0$. *If* $x > 1$ *then* $\zeta(x) > 1$.

PROOF. Recall that $\zeta(s) = s/(s-1) - s\int_1^{\infty} \{t\}dt/t^{s+1}$ from the proof of Lemma 4.2. If $0 < x < 1$ then all these terms are negative. If $x > 1$ then all terms in the Dirichlet series are positive, and the first term is 1. $\square$

COROLLARY 4.4 (Ramanujan's divergent series estimate).

$$\zeta(-1) = \frac{-1}{12}.$$

PROOF. Immediate from Lemma 4.2 by substituting $s = -1$. $\square$

LEMMA 4.5 (Growth of $\zeta(s)$). *For* $\Re(s) \geq -19/10$ *and* $|s - 1| \geq 1$ *we have*

$$|\zeta(s)| = O(1 + |s|^3).$$

PROOF. From Lemma 4.2, for $\Re(s) \geq -19/10$ and $|s - 1| \geq 1$ we have

$$|\zeta(s)| = O(1) + O(|s|) + \int_1^{\infty} \frac{O(|s|^3)}{t^{\Re(s)+3}} dt = O(1 + |s|^3). \qquad \square$$

EXAMPLE 4.6 (Dirichlet Series for $\zeta'(s)$). *For* $\Re(s) > 1$ *we have*

$$\sum_{n=1}^{\infty} \frac{\log n}{n^s} = -\zeta'(s).$$

Proof. Let $\Re(s) = 1 + \delta$. By the maximum modulus principle

$$\sup_{|z|\le\delta/2}\Big|\frac{1}{z}\Big(\frac{1}{n^{s+z}} - \frac{1}{n^s}\Big)\Big| = \sup_{|z|=\delta/2}\Big|\frac{1}{z}\Big(\frac{1}{n^{s+z}} - \frac{1}{n^s}\Big)\Big| \le \frac{4}{\delta n^{1+\delta/2}}.$$

This converges when summed over n, so by the dominated convergence theorem

$$\begin{aligned}\zeta'(s) = \lim_{z\to 0}\Big(\frac{\zeta(s+z) - \zeta(s)}{z}\Big) &= \lim_{z\to 0}\lim_{N\to\infty}\sum_{n=1}^{N}\Big(\frac{1}{zn^{s+z}} - \frac{1}{zn^s}\Big)\\ &= \lim_{N\to\infty}\lim_{z\to 0}\sum_{n=1}^{N}\Big(\frac{1}{zn^{s+z}} - \frac{1}{zn^s}\Big)\\ &= -\sum_{n=1}^{\infty}\frac{\log n}{n^s}.\end{aligned}$$

□

Lemma 4.7 (Approximate formula for $\zeta(s)$). *Let $s = \sigma + it$ with $\sigma > 0$ and let $N \in \mathbb{Z}_{>0}$. Then we have*

$$\zeta(s) = \sum_{n=1}^{N}\frac{1}{n^s} + \frac{N^{1-s}}{s-1} + O(|s|N^{-\sigma}).$$

Proof. We follow the proof of Lemma 4.2, but only working with the terms bigger than N. For $\sigma > 1$

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{N}\frac{1}{n^s} + \sum_{n=N+1}^{\infty}\frac{1}{n^s}\\ &= \sum_{n=1}^{N}\frac{1}{n^s} + s\int_N^\infty \frac{t - N - \{t-N\}}{t^{s+1}}\\ &= \sum_{n=1}^{N}\frac{1}{n^s} + \frac{N^{1-s}}{s-1} + s\int_N^\infty \frac{-\{t-N\}}{t^{s+1}}.\end{aligned}$$

Since the integrand is $O(t^{-1-\sigma})$, the right hand side converges for all $\sigma > 0$ and the final integral is $O(|s|N^{-\sigma})$ throughout this region. □

Corollary 4.8 (Growth of $\zeta(s)$, II). *For $0 < \sigma < 1$ we have*

$$|\zeta(\sigma + it)| \ll 1 + \frac{|t|^{1-\sigma}}{1-\sigma}.$$

Proof. We use the above lemma. We see that

$$\begin{aligned}\Big|\sum_{n=1}^{N}\frac{1}{n^s}\Big| \le 1 + \sum_{n=2}^{N}\frac{1}{n^\sigma} \le 1 + \int_1^N \frac{du}{u^\sigma} &\ll \frac{N^{1-\sigma}}{1-\sigma}\\ \Big|\frac{N^{1-s}}{s-1}\Big| &\le \frac{N^{1-\sigma}}{1-\sigma}\\ |s|N^{-\sigma} \le N^{-\sigma} &+ |t|N^{-\sigma}\end{aligned}$$

Choosing $N = \lceil t \rceil$ then gives the result. □

CHAPTER 5

Euler Products

In this section we make use of the key observation of Euler; that a Dirichlet series $\sum_n f(n)n^{-s}$ has a product representation if f has the special property of being multiplicative.

LEMMA 5.1. *Let f be a multiplicative function with $|f(n)| \leq n^{o(1)}$. Then for $\Re(s) > 1$ we have*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \Big(1 + \sum_{j=1}^{\infty} \frac{f(p^j)}{p^{js}}\Big).$$

In particular, if f is completely multiplicative then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \Big(1 - \frac{f(p)}{p^s}\Big)^{-1}.$$

PROOF. We first want to show that the finite expressions

$$S_{M_1,M_2}(s) = \prod_{p<M_1} \Big(1 + \sum_{j=1}^{M_2} \frac{f(p^j)}{p^{js}}\Big)$$

converge to the Dirichlet series $\sum_{n=1}^{\infty} f(n)n^{-s}$ as $M_2 \to \infty$ and then $M_1 \to \infty$.

If an integer n has a prime factorization $n = p_1^{e_1} \dots p_j^{e_j}$, then, since f is multiplicative, we see that

$$\frac{f(n)}{n^s} = \frac{f(p_1^{e_1})}{p_1^{e_1 s}} \cdot \frac{f(p_2^{e_2})}{p_2^{e_2 s}} \cdots \frac{f(p_j^{e_j})}{p_j^{e_j s}}.$$

Therefore, if we expand the product $S_{M_1,M_2}(s)$, we find that

$$S_{M_1,M_2}(s) = \sum_{n \in \mathcal{N}} \frac{f(n)}{n^s}$$

where $\mathcal{N}$ is the (finite) set of all integers n whose prime factorization only involves primes $p < M_1$, and each such prime occurs at most M_2 times in the prime factorization. Note that here we have made crucial use of the unique factorization of integers. We see that $\mathcal{N}$ certainly contains all integers of size at most $M = \min(M_1, M_2)$. Therefore we see that for $M_1 \leq M_2$

$$\Big|S_{M_1,M_2}(s) - \sum_{n=1}^{\infty} \frac{f(n)}{n^s}\Big| \leq \sum_{n \notin \mathcal{N}} \frac{|f(n)|}{|n^s|} \leq \sum_{n > M_1} \frac{|f(n)|}{|n^s|}$$

Letting $M_2 \to \infty$, this gives

$$\Big|\prod_{p<M_1}\Big(1+\sum_{j=1}^{\infty}\frac{f(p^j)}{p^{js}}\Big)-\sum_{n=1}^{\infty}\frac{f(n)}{n^s}\Big| \le \sum_{n>M_1}\frac{|f(n)|}{|n^s|}$$

But since the series $\sum_{n=1}^{\infty} f(n)n^{-s}$ converges absolutely, the right hand side tends to 0 as M_1 tends to infinity, which then gives the first result.

If f is completely multiplicative then $f(p^j) = f(p)^j$ so the sum is a geometric series, which simplifies to the expression given. □

COROLLARY 5.2 (Euler product for $\zeta(s)$).

$$\zeta(s) = \prod_p \Big(1-\frac{1}{p^s}\Big)^{-1}$$

PROOF. This is just Lemma 5.1 applied to $f(n) = 1$, a completely multiplicative function. □

LEMMA 5.3 (Dirichlet series of convolution is product of Dirichlet series). *Let $f,\ g : \mathbb{Z}_{>0} \to \mathbb{C}$. Then*

$$\sum_{n=1}^{\infty}\frac{(f\star g)(n)}{n^s} = \Big(\sum_{a=1}^{\infty}\frac{f(a)}{a^s}\Big)\Big(\sum_{b=1}^{\infty}\frac{g(b)}{b^s}\Big)$$

whenever s is such that $\sum_{a=1}^{\infty} f(a)a^{-s}$ and $\sum_{b=1}^{\infty} g(b)b^{-s}$ converge absolutely.

PROOF. We have

$$\sum_{n=1}^{N}\frac{(f\star g)(n)}{n^s} = \sum_{n=1}^{N}\frac{\sum_{ab=n} f(a)g(b)}{n^s} = \sum_{\substack{a,b\\ ab\le N}}\frac{f(a)g(b)}{a^s b^s}.$$

If $ab \le N$ then either $a > N^{1/2}$ or $b > N^{1/2}$ or both are at most $N^{1/2}$. Thus

$$\Big|\sum_{n=1}^{N}\frac{(f\star g)(n)}{n^s}-\Big(\sum_{a=1}^{N^{1/2}}\frac{f(a)}{a^s}\Big)\Big(\sum_{b=1}^{N^{1/2}}\frac{g(b)}{b^s}\Big)\Big|$$
$$\le \sum_{N^{1/2}<a\le N}\Big|\frac{f(a)}{a^s}\Big|\sum_{b<N}\Big|\frac{g(b)}{b^s}\Big| + \sum_{a\le N}\Big|\frac{f(a)}{a^s}\Big|\sum_{N^{1/2}<b\le N}\Big|\frac{g(b)}{b^s}\Big|$$

Since, by assumption, the series $\sum_a f(a)a^{-s}$ and $\sum_b g(b)b^{-s}$ converge absolutely, the right hand side tends to zero as $N \to \infty$, and the Dirichlet series of the convolution converges to the product of the Dirichlet series. □

LEMMA 5.4 (Dirichlet Series for ζ^2, $1/\zeta$ and ζ'/ζ). *For* $\Re(s) > 1$ *we have*

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \zeta(s)^2,$$

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)},$$

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)}.$$

PROOF. We observe that $\tau = 1 \star 1$, and so the first result follows from Lemma 5.3.

Since $|\mu(n)| \leq 1$ and μ is multiplicative, we have that for $\Re(s) > 1$

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \Big(1 + \sum_{j=1}^{\infty} \frac{\mu(p^j)}{p^{js}}\Big) = \prod_p \Big(1 - \frac{1}{p^s}\Big),$$

and that both sides converge absolutely. But clearly the right hand side is the Euler product for $1/\zeta(s)$.

Since $\Lambda = \mu \star \log$, we see that the second part follows from Lemma 5.3 and Lemma 4.6. □

COROLLARY 5.5 (Non-vanishing of $\zeta(s)$ in $\Re(s) > 1$). *If* $\Re(s) > 1$ *then*

$$\zeta(s) \neq 0.$$

PROOF. By Lemma 5.4, for $\Re(s) > 1$ we have

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

which converges absolutely. Thus we cannot have $\zeta(s) = 0$ in this region. □

CHAPTER 6

Poisson Summation

Definition (Schwartz spaces). *Let $\mathcal{S}(\mathbb{R}/\mathbb{Z})$ be the space of all infinitely differentiable functions $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$.*

Let $\mathcal{S}(\mathbb{Z})$ be the space of all functions $f : \mathbb{Z} \to \mathbb{C}$ such that for every $k \in \mathbb{Z}_{>0}$ we have $f(x) = O_k(|x|^{-k})$.

Let $\mathcal{S}(\mathbb{R})$ be the space of all infinitely differentiable functions $f : \mathbb{R} \to \mathbb{C}$ such that for every $k, j \in \mathbb{Z}_{>0}$ we have $f^{(j)}(x) = O_{k,j}(|x|^{-k})$.

Lemma 6.1 (Fourier transform for $\mathcal{S}(\mathbb{R})$). *Let $f \in \mathcal{S}(\mathbb{R})$. Then the Fourier transform*

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$$

is a function in $\mathcal{S}(\mathbb{R})$.

Proof. First note that since $f \in \mathcal{S}(\mathbb{R})$, we have that $f(x) = O(|x|^{-k})$ for $|x| \geq 1$. Thus $\hat{f}(\xi)$ is given by an absolutely convergent integral, and

$$\frac{\hat{f}(\xi+\epsilon) - \hat{f}(\xi)}{\epsilon} = \int_{|x|<\epsilon^{-1/2}} f(x) e^{-2\pi i x \xi} \Big(\frac{e^{-2\pi i x \epsilon} - 1}{\epsilon} \Big) dx + \int_{|x| \geq \epsilon^{-1/2}} O\Big(\frac{|f(x)|}{\epsilon} \Big) dx.$$

In the first integral we use the Taylor expansion $e^{-2\pi i x \epsilon} = 1 - 2\pi i x \epsilon + O(x^2 \epsilon^2)$. Thus, taking out a term $-2\pi i x f(x) e^{-2\pi i x \xi}$ from both integrals, we find

$$\begin{aligned}
\frac{\hat{f}(\xi+\epsilon) - \hat{f}(\xi)}{\epsilon} &= \int_{-\infty}^{\infty} -2\pi i x f(x) e^{-2\pi i x \xi} + O\Big(\int_{|x|<\epsilon^{-1/2}} \epsilon x^2 |f(x)| dx \Big) \\
&\qquad + O\Big(\int_{|x| \geq \epsilon^{-1/2}} |f(x)| (\frac{1}{\epsilon} + x) dx \Big) \\
&= \int_{-\infty}^{\infty} -2\pi i x f(x) e^{-2\pi i x \xi} + O\Big(\int_{|x|<\epsilon^{-1/2}} \epsilon dx \Big) + O\Big(\int_{|x| \geq \epsilon^{-1/2}} \frac{1}{x^4 \epsilon} + \frac{1}{x^3} \Big) dx \Big) \\
&= \int_{-\infty}^{\infty} -2\pi i x f(x) e^{-2\pi i x \xi} + O(\epsilon^{1/2}).
\end{aligned}$$

This converges as $\epsilon \to 0$, showing $\hat{f}'(\xi)$ is the Fourier transform of $-2\pi i x f(x)$. Since $-2\pi i x f(x) \in \mathcal{S}(\mathbb{R})$ whenever $f \in \mathcal{S}(\mathbb{R})$, we can repeat the above argument and find that $\hat{f}^{(j)}$ is the Fourier transform of $(-2\pi i x)^j f(x)$ for all $j \in \mathbb{Z}_{>0}$.

By differentiating by parts k times, we see that

$$\hat{f}^{(j)}(\xi) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{(2\pi i\xi)^k} \frac{\partial^k}{\partial x^k}\Big((-2\pi i x)^j f(x)\Big) dx \ll_{j,k} \frac{1}{|\xi|^k}.$$

Thus $\hat{f} \in \mathcal{S}(\mathbb{R})$. □

LEMMA 6.2 (Fourier inversion for $\mathcal{S}(\mathbb{R}/\mathbb{Z})$). *Let $g \in \mathcal{S}(\mathbb{R}/\mathbb{Z})$. Then the Fourier transform*

$$\hat{g}(n) = \int_0^1 g(\theta) e^{-2\pi i n\theta} d\theta$$

is a function in $\mathcal{S}(\mathbb{Z})$ such that

$$g(\theta) = \sum_{n\in\mathbb{Z}} \hat{g}(n) e^{2\pi i n\theta}.$$

PROOF. Since g is infinitely differentiable, by integration by parts we see that

$$\hat{g}(n) = \int_0^1 g^{(j)}(\theta) \frac{e^{2\pi i n\theta}}{(-2\pi i n)^j} d\theta \ll_j \frac{1}{|n|^j}.$$

Thus $\hat{g} \in \mathcal{S}(\mathbb{Z})$. Let $h \in \mathcal{S}(\mathbb{R}/\mathbb{Z})$ be given by

$$h(\theta) = g(\theta) - \sum_{n\in\mathbb{Z}} \hat{g}(n) e^{2\pi i n\theta}.$$

We want to show that $h(\theta) = 0$. Assume for a contradiction that $h(\theta_1) \neq 0$ for some θ_1. We first see that

$$\hat{h}(m) = \int_0^1 g(\theta) e^{-2\pi i m\theta} d\theta - \sum_{n\in\mathbb{Z}} \hat{g}(n) \int_0^1 e^{2\pi i (n-m)\theta} d\theta = \hat{g}(m) - \hat{g}(m) = 0.$$

Similarly we see that all the Fourier coefficients of $\overline{h}$ vanish. Thus if $h \neq 0$, by considering $f = \pm(h + \overline{h})$ or $f = \pm(h - \overline{h})/i$, we see there exists a real function $f \in \mathcal{S}(\mathbb{R}/\mathbb{Z})$ with $f(\theta_1) > 0$ but $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$.

Since $f(\theta_1) > 0$, there is an $\epsilon > 0$ such that $f(\theta) > f(\theta_1)/2$ for $\theta \in [\theta_1 - \epsilon, \theta_1 + \epsilon]$. Then there is a $\delta > 0$ such that $|\cos(2\pi(\theta - \theta_1)) + \delta| < 1 - \delta/2$ for all $\theta \notin [\theta_1 - \epsilon, \theta_1 + \epsilon]$, and there is an $\eta > 0$ such that $\eta < \epsilon$ and $\cos(2\pi i(\theta - \theta_1)) + \delta > 1 + \delta/2$ for all $\theta \in [\theta_1 - \eta, \theta_1 + \eta]$.

Consider the function $(\delta + \cos(2\pi(\theta - \theta_1)))^k$ for some large integer k. This can be expanded as trigonometric polynomial $\sum_{-k\le j\le k} c_j e^{2\pi i j\theta}$ for some coefficients c_j. Since all Fourier coefficients of f vanish, we see that

$$\int_0^1 \Big(\delta + \cos(2\pi(\theta - \theta_1))\Big)^k f(\theta) d\theta = \sum_{-k\le j\le k} c_j \int_0^1 f(\theta) e^{2\pi i j\theta} d\theta = 0.$$

On the other hand, this integral is given by

$$\begin{aligned}&\int_{|\theta-\theta_1|\leq\epsilon} f(\theta)(\delta+\cos(2\pi(\theta-\theta_1)))^k + \int_{|\theta-\theta_1|\geq\epsilon} O(1-\delta/2)^k d\theta\\ &\geq \int_{|\theta-\theta_1|\leq\eta} f(\theta)(\delta+\cos(2\pi(\theta-\theta_1)))^k + O(1-\delta/2)^k\\ &\geq 2\eta\frac{f(\theta_1)}{2}(1+\delta/2)^k + O(1-\delta/2)^k.\end{aligned}$$

Thus for k large enough the integral is non-zero, giving a contradiction. □

THEOREM 6.3 (Poisson summation formula). *Let $f \in \mathcal{S}(\mathbb{R})$ with Fourier transform $\hat{f}$. Then*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m).$$

PROOF. Define two functions $F, G : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ by

$$\begin{aligned}F(\theta) &= \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi in\theta},\\ G(\theta) &= \sum_{m\in\mathbb{Z}} f(\theta+m).\end{aligned}$$

Since $f \in \mathcal{S}(\mathbb{R})$ and $\hat{f} \in \mathcal{S}(\mathbb{R})$ by Lemma 6.1, it is easy to verify that $F, G \in \mathcal{S}(\mathbb{R}/\mathbb{Z})$. We want to show that $F = G$. We do this by computing Fourier coefficients. For $m \in \mathbb{Z}$, we find

$$\begin{aligned}\hat{F}(m) &= \int_0^1 \Big(\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi in\theta}\Big)e^{-2\pi im\theta}d\theta\\ &= \sum_{n\in\mathbb{Z}} \hat{f}(n)\int_0^1 e^{2\pi i(n-m)\theta}d\theta = \hat{f}(m).\end{aligned}$$

Similarly

$$\begin{aligned}\hat{G}(m) &= \int_0^1 \Big(\sum_{n\in\mathbb{Z}} f(\theta+n)\Big)e^{-2\pi im\theta}d\theta\\ &= \sum_{n\in\mathbb{Z}}\int_n^{n+1} f(\theta)e^{-2\pi im(\theta-n)}d\theta\\ &= \sum_{n\in\mathbb{Z}}\int_n^{n+1} f(\theta)e^{-2\pi im\theta}d\theta = \hat{f}(m).\end{aligned}$$

(We may exchange the orders of summation and integration above since $f, \hat{f} \in \mathcal{S}(\mathbb{R})$ so everything converges absolutely.)

By Lemma 6.2, F and G are uniquely determined by their Fourier coefficients, and so are equal. □

LEMMA 6.4 (Meromorphic continuation of modified $\zeta(s)$). *Let $f \in \mathcal{S}(\mathbb{R})$ satisfy $f(x) = f(-x)$. Define the Mellin transform*

$$F(s) = \int_0^\infty f(x)x^{s-1}dx.$$

Then we have

$$\zeta(s)F(s) = \frac{\hat{f}(0)}{2s-2} - \frac{f(0)}{2s} + \int_1^\infty \Big(\sum_{n=1}^\infty f(nx)\Big)x^{s-1}dx + \int_1^\infty \Big(\sum_{n=1}^\infty \hat{f}(nu)\Big)u^{-s}du,$$

and the right hand side converges to a meromorphic function for all $s \in \mathbb{C}$ with poles at $s = 0$ and $s = 1$.

PROOF. For $\Re(s) > 1$ we have

$$\zeta(s)F(s) = \sum_{n=1}^\infty \frac{1}{n^s}\int_0^\infty f(x)x^{s-1}dx = \sum_{n=1}^\infty \int_0^\infty f(nt)t^{s-1}dt = \int_0^\infty \Big(\sum_{n=1}^\infty f(nt)\Big)t^{s-1}dt.$$

(We may exchange the order of summation and integration using Fubini's Theorem since $f \in \mathcal{S}(\mathbb{R})$ and so everything coverges absolutely.) Let $h(x) = f(xt)$. Then

$$\hat{h}(\xi) = \int_{-\infty}^\infty f(xt)e^{-2\pi i x\xi}dx = \frac{1}{t}\int_{-\infty}^\infty f(u)e^{-2\pi i u\xi/t}du = \frac{\hat{f}(\xi/t)}{t}.$$

Thus, by Poisson summation (Theorem 6.3) we have

$$\sum_{n\in\mathbb{Z}} f(nt) = \sum_{n\in\mathbb{Z}} h(n) = \sum_{m\in\mathbb{Z}} \hat{h}(m) = \frac{1}{t}\sum_{m\in\mathbb{Z}} \hat{f}\Big(\frac{m}{t}\Big).$$

If f is even, then $\hat{f}$ is also even, so we find

$$\sum_{n=1}^\infty f(nt) = \frac{1}{2}\sum_{n\in\mathbb{Z}} f(nt) - \frac{f(0)}{2} = \frac{\hat{f}(0)}{2t} - \frac{f(0)}{2} + \frac{1}{t}\sum_{m=1}^\infty \hat{f}\Big(\frac{m}{t}\Big).$$

We separate the integral expression for $\zeta(s)F(s)$ into $0 \le x \le 1$ and $1 \le x \le \infty$, and substitute the above expression into the integral over $0 \le x \le 1$. This gives for $\Re(s) > 1$

$$\begin{aligned}
\zeta(s)F(s) &= \int_1^\infty \Big(\sum_{n=1}^\infty f(nt)\Big)t^{s-1}dt + \int_0^1 \Big(\sum_{n=1}^\infty f(nt)\Big)t^{s-1}dt \\
&= \int_1^\infty \Big(\sum_{n=1}^\infty f(nt)\Big)t^{s-1}dt + \frac{\hat{f}(0)}{2s-2} - \frac{f(0)}{2s} + \int_0^1 \Big(\sum_{m=1}^\infty \hat{f}\Big(\frac{m}{t}\Big)\Big)t^{s-2}dt \\
&= \int_1^\infty \Big(\sum_{n=1}^\infty f(nt)\Big)t^{s-1}dt + \frac{\hat{f}(0)}{2s-2} - \frac{f(0)}{2s} + \int_1^\infty \Big(\sum_{m=1}^\infty \hat{f}(mu)\Big)u^{-s}du,
\end{aligned}$$

where in the final integral on the final line we substituted $u = 1/t$. □

CHAPTER 7

The Functional Equation

Lemma 7.1 (Gaussian is eigenfunction of Fourier operator). *Let $f(x) = e^{-\pi x^2} \in \mathcal{S}(\mathbb{R})$. Then*

$$\hat{f}(\xi) = e^{-\pi \xi^2}.$$

Proof. By completing the square, we have

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-\pi x^2 - 2\pi i x \xi} dx = e^{-\pi \xi^2} \int_{-\infty}^{\infty} e^{-\pi (x+i\xi)^2} dx.$$

By Cauchy's residue theorem

$$\int_{-R+i\xi}^{R+i\xi} f(z)dz + \int_{R+i\xi}^{R} f(z)dz + \int_{R}^{-R} f(z)dz + \int_{-R}^{-R+i\xi} f(z)dz = 0,$$

where the integrals are straight line contours. Since $|f(z)| \le e^{-\pi(\Re(z)^2 - \Im(z)^2)}$, we see that the second and fourth terms both tend to 0 as $R \to \infty$. Thus we find that

$$\int_{-\infty}^{\infty} e^{-\pi(x+i\xi)^2} dx = \lim_{R\to\infty} \int_{-R+i\xi}^{R+i\xi} f(z)dz = -\int_{\infty}^{-\infty} f(z)dz = \int_{-\infty}^{\infty} e^{-\pi x^2} dx.$$

The result follows on recalling the identity $\int_{-\infty}^{\infty} e^{-\pi u^2} du = 1$. □

Definition (The Gamma function). *For $\Re(s) > 0$, let $\Gamma(s)$ be defined by*

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx.$$

Theorem 7.2 (The functional equation). *Define the function*

$$\xi(s) = \pi^{-s/2} \Gamma\Big(\frac{s}{2}\Big) \zeta(s).$$

Then $\xi(s)$ has a meromorphic continuation to the entire complex plane, and satisfies the functional equation

$$\xi(s) = \xi(1-s).$$

Proof. We apply Lemma 6.4 with $f(x) = e^{-\pi x^2}$. We see that substituting $y = \pi x^2$ gives

$$F(s) = \int_0^{\infty} e^{-\pi x^2} x^{s-1} dx = \frac{1}{2\pi^{s/2}} \int_0^{\infty} e^{-y} y^{s/2-1} dy = \frac{\Gamma(s/2)}{2\pi^{s/2}}.$$

Thus Lemma 6.4 gives

$$\frac{\Gamma(s/2)\zeta(s)}{2\pi^{s/2}} = \frac{1}{2s-2} - \frac{1}{2s} + \int_1^\infty \Big(\sum_{n=1}^\infty e^{-\pi n^2 x^2}\Big) x^{s-1} dx + \int_1^\infty \Big(\sum_{n=1}^\infty e^{-\pi n^2 x^2}\Big) x^{-s} dx,$$

and we see that the right hand side is unchanged if we replace s with $1-s$. □

Lemma 7.3 (Functional equation for $\Gamma(s)$). *For $\Re(s) > 0$, we have*

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}$$

Proof. This is integration by parts:

$$\begin{aligned}\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx &= \Big[\frac{x^s}{s} \cdot -e^{-x}\Big]_0^\infty - \int_0^\infty \frac{x^s}{s}\Big(-e^{-x}\Big) dx \\ &= \frac{\Gamma(s+1)}{s}.\end{aligned}$$

□

Corollary 7.4. *$\Gamma(s)$ has a meromorphic continuation to the entire complex plane, with poles only at the non-positive integers.*

Proof. From Lemma 7.3 see that $\Gamma(s+1)/s$ is a meromorphic continuation of $\Gamma(s)$ to the region $\Re(s) > -1$ with a simple pole at $s = 0$. By repeatedly applying Lemma 7.3, we see that for any $n \in \mathbb{Z}_{>0}$ we have

$$\Gamma(s) = \frac{\Gamma(s+n)}{s(s+1)\cdots(s+n-1)},$$

and this defines an analytic continuation of $\Gamma(s)$ to $\Re(s) > -n$, with possible poles only at $s = 0, -1, -2, \dots, -(n-1)$. This gives the result. □

Corollary 7.5. *$\zeta(s)$ has a meromorphic continuation to the entire complex plane.*

Proof. $\zeta(s) = \xi(s)\pi^{s/2}/\Gamma(s/2)$, and the right hand side has a suitable continuation. □

Lemma 7.6 (Euler reflection fomula). *For all $s \in \mathbb{C}$ we have*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

Proof. We see that $\Gamma(s)\Gamma(1-s)$ is a meromorphic function which has at most a simple pole at $s \in \mathbb{Z}$ and no other poles. Since $\sin(\pi s) = (e^{i\pi s} - e^{-i\pi s})/2$ has zeros at $n \in \mathbb{Z}$ and has no poles, we see $G(s) = \Gamma(s)\Gamma(1-s)\sin(\pi s)$ is an entire function. Moreover, $G(s) = G(s+1)$, so we can define an analytic function $F(s)$ on $\mathbb{C}\backslash\{0\}$ by $F(Re^{i\theta}) := G((\theta - i\log R)/2\pi)$. By Lemma 7.3 we have

$$|G(\sigma+it)| = \Big|\frac{\Gamma(1+\sigma+it)\Gamma(2-\sigma+it)}{(\sigma+it)(1-\sigma+it)}\Big| \cdot \Big|\frac{e^{-\pi t + i\pi\sigma} + e^{\pi t - i\pi\sigma}}{2}\Big|.$$

For $0 < \sigma$, the integral equation shows that $|\Gamma(\sigma + it)| \le |\Gamma(\sigma)|$. Therefore for $|t| > 1$ and $0 \le \sigma \le 1$ we have

$$|G(\sigma + it)| \le \frac{\Gamma(1+\sigma)}{|\sigma + it|}\frac{\Gamma(2-\sigma)}{|1-\sigma+it|}e^{\pi|t|} \ll e^{\pi|t|}.$$

Thus for $R < e^{-2\pi}$ or $R > e^{2\pi}$ we have

$$|F(Re^{i\theta})| = |G((\theta - i\log R)/2\pi)| \ll R^{1/2} + R^{-1/2}.$$

In particular, since $sF(s) \to 0$ as $s \to 0$, we can extend $F(s)$ to an entire function on all of $\mathbb{C}$. But then for large R

$$\begin{aligned} F(s) - F(0) &= \int_{|z|=R} \Big(\frac{F(z)}{z-s} - \frac{F(z)}{z}\Big) dz \\ &= s\int_{|z|=R} \frac{F(z)}{z(z-s)} dz \\ &\ll |s| \int_{|z|=R} O(R^{-3/2})|dz| \ll \frac{|s|}{R^{1/2}}. \end{aligned}$$

Since this is true for all large R, letting $R \to \infty$ shows that F is constant. Thus G is a constant. To find the value of the constant, we see that as $s \to 0$ we have

$$G(s) = \Gamma(1+s)\Gamma(1-s)\frac{\sin(\pi s)}{s} \to \Gamma(1)^2\pi = \pi. \qquad \square$$

COROLLARY 7.7 (Non-vanishing of $\Gamma(s)$). *$\Gamma(s)$ has no zeros.*

PROOF. $\sin(\pi s)$ has no poles, so this follows immediately from Lemma 7.6. $\square$

LEMMA 7.8 (Zeros and poles of $\zeta(s)$). *$\zeta(s)$ is a meromorphic function with*

- *A simple pole at $s = 1$, and no other poles.*
- *'Trivial zeros' at $s = -2, -4, \dots,$ and no other zeros in $\Re(s) < 0$.*
- *'Non-trivial' zeros ρ with $\Re(\rho) \in [0, 1]$.*
- *No zeros in $\Re(s) > 1$.*

PROOF. The functional equation (Theorem 7.2) gives

$$\zeta(s) = \frac{\pi^{s/2}\Gamma\Big(\frac{1-s}{2}\Big)}{\pi^{(1-s)/2}\Gamma(s/2)}\zeta(1-s).$$

$\zeta(1-s)$ has no zeros in $\Re(s) < 0$ by Corollary 5.5, and in $\Re(s) \le 1$ it has only a simple pole at $s = 0$ by Lemma 4.2. $\pi^{s/2}$ and $\pi^{(1-s)/2}$ have no zeros or poles in the complex plane. $\Gamma((1-s)/2)$ has no zeros by Corollary 7.7 and has a unique simple poles at $s = 1$ in $\Re(s) \le 1$. $\Gamma(s/2)$ has simple poles at $s = 0, -2, -4, \dots$ and no zeros. Putting these statements together, we see that the right hand side has a removable singularity at $s = 0$, a simple pole at $s = 1$ and no other poles in $\Re(s) \le 1$. Moreover, it has zeros at $s = -2, -4, \dots$ but no other zeros in $\Re(s) < 0$. Recalling $\zeta(s)$ has no zeros in $\Re(s) > 1$ by Corollary 5.5 gives the result. $\square$

CHAPTER 8

Perron's Formula

LEMMA 8.1. *Let $y > 0$ and $y \neq 1$. Then for any $c > 0$ and $T \geq 2$ we have*

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{y^s ds}{s} = H(y) + O\Big(\frac{y^c}{T|\log y|}\Big),$$

where

$$H(y) = \begin{cases} 1, & \text{if } y > 1, \\ 0, & \text{if } y < 1. \end{cases}$$

PROOF. This is an exercise in Cauchy's residue theorem. The integrand y^s/s is meromorphic in the whole complex plane with a simple zero at $s = 0$ with residue 1. If $y > 1$ then the residue theorem implies that for any $r > 1$

$$\int_{c-iT}^{c+iT}\frac{y^s ds}{s} + \int_{c+iT}^{-r+iT}\frac{y^s ds}{s} + \int_{-r+iT}^{-r-iT}\frac{y^s ds}{s} + \int_{-r-iT}^{c-iT}\frac{y^s ds}{s} = 2\pi i \operatorname*{Res}_{s=0}\frac{y^s}{s} = 2\pi i.$$

The first term on the left hand side is the thing we want to estimate. In the second and fourth integrals we have $|s| \geq T$ and $|y^s| \leq y^{\Re(s)}$, so they are each bounded in size by

$$\int_{-r}^{c}\frac{y^\sigma d\sigma}{T} \leq \frac{1}{T}\int_{-\infty}^{c} y^\sigma d\sigma = \frac{y^c}{T|\log y|}.$$

In the third integral we have $|y^s| \leq y^{-r}$ and $|s| \geq r$, so this is bounded by

$$\int_{-T}^{T}\frac{y^{-r}dt}{r} \leq \frac{2y^{-r}T}{r}.$$

Putting this together, we see that

$$\int_{c-iT}^{c+iT}\frac{y^s ds}{s} = 2\pi i + O\Big(\frac{y^c}{T\log y}\Big) + O\Big(\frac{y^{-r}T}{r}\Big).$$

Letting $r \to \infty$ then gives the result in this case. If instead $y < 1$, then we apply the same argument but with $r < 0$. In this case the closed contour avoids the pole at $s = 0$, and so we find the the same argument gives

$$\int_{c-iT}^{c+iT}\frac{y^s ds}{s} = O\Big(\frac{y^c}{T|\log y|}\Big) + O\Big(\frac{y^{-r}T}{|r|}\Big).$$

Letting $r \to -\infty$ gives the result. ☐

LEMMA 8.2 (Perron's formula). *Let* $2 \le T \le 2x$ *and* $c = 1 + 1/\log x$. *Let* $a_n \in \mathbb{C}$ *be a complex sequence with* $|a_n| \le (\log n)^2$. *Then*

$$A(x) = \sum_{n<x} a_n = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} x^s \Big(\sum_{n=1}^{\infty} \frac{a_n}{n^s}\Big)\frac{ds}{s} + O\Big(\frac{x(\log x)^3}{T}\Big).$$

PROOF. We note that if $|n-x| \le 3$ and $n \ge 1$ then $(x/n)^c = O(1)$, so

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \Big(\frac{x}{n}\Big)^s \frac{ds}{s} \ll \log T \le \log x.$$

Using this if $|n-x| \le 3$ and using Lemma 8.1 if $|n-x| > 3$, we have for any $N > x$

$$\begin{aligned}\sum_{n<x} a_n &= \sum_{\substack{n<N\\ |n-x|>3}} a_n\Big(\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{x^s ds}{n^s s} + O\Big(\frac{(x/n)^c}{T|\log(x/n)|}\Big)\Big)\\ &\quad + \sum_{|n-x|\le 3} a_n\Big(\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{x^s ds}{n^s s} + O(\log T)\Big)\\ &= \frac{1}{2\pi i}\int_{c-iT}^{c+iT} x^s\Big(\sum_{n=1}^{N}\frac{a_n}{n^s}\Big)\frac{ds}{s} + O\Big(\frac{x^c}{T}\sum_{\substack{n<N\\ |n-x|\ge 3}} \frac{(\log n)^2}{n^c|\log(x/n)|}\Big) + O(\log x)^3.\end{aligned}$$

We first concentrate on the error term. We note that $x^c \ll x$. If $n < 3x/4$ then $|\log(x/n)| \gg \log(4/3) > 0$. Thus these terms contribute

$$\ll \frac{x}{T}\sum_{n<3x/4}\frac{(\log n)^2}{n^c} \ll \frac{x}{T}\sum_{n<3x/4}\frac{(\log n)^2}{n} \ll \frac{x(\log x)^3}{T}.$$

Similarly, if $n > 5x/4$ then $|\log(x/n)| \ge \log(5/4) > 0$, and so these terms contribute

$$\ll \frac{x}{T}\sum_{n>5x/4}\frac{(\log n)^2}{n^c} \ll \frac{x}{T}\int_{5x/4-1}^{\infty}\frac{(\log t)^2 dt}{t^{1+1/\log x}} \ll \frac{x(\log x)^3}{T}.$$

For the terms $3x/4 \le n \le 5x/4$ we put $n = \lfloor x \rfloor + h$ and note that $|\log(n/x)| = |\log(1 + (n-x)/x)| \ge |h|/2x$ for $3 \le |h| \le x/4$. Thus these terms contribute

$$\ll \frac{x}{T}\sum_{3\le |h|\le x/4}\frac{(\log x)^2 x}{|h|(\lfloor x\rfloor + h)^c} \ll \frac{x(\log x)^2}{T}\sum_{1\le h\le x/4}\frac{1}{h} \ll \frac{x(\log x)^3}{T}.$$

Putting this together, we find that for any $N > x$ we have

$$\sum_{n<x} a_n = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} x^s\Big(\sum_{n=1}^{N}\frac{a_n}{n^s}\Big)\frac{ds}{s} + O\Big(\frac{x(\log x)^3}{T}\Big).$$

Since the Dirichlet series converges uniformly absolutely on $\Re(s) \ge 1+\delta$ by Lemma 4.1, letting $N \to \infty$ gives the result. □

LEMMA 8.3 (Counting primes). *Let $c = 1 + 1/\log x$ and $2 \le T \le 2x$. Then we have*

$$\psi(x) = \sum_{n<x} \Lambda(n) = \frac{-1}{2\pi i} \int_{c-iT}^{c+iT} x^s \frac{\zeta'}{\zeta}(s) \frac{ds}{s} + O\Big(\frac{x(\log x)^3}{T}\Big).$$

PROOF. This follows immediately from Lemma 5.4 and Lemma 8.2, noting that $\Lambda(n) \le \log n$. □

CHAPTER 9

$\zeta(s)$ as a Taylor Series

LEMMA 9.1 (Taylor coefficients are controlled by size of function). *Let $f(z)$ be an analytic function on the disk $|z| \le R$, with Taylor series $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Then for $n \ge 1$ the coefficients c_n satisfy*

$$|c_n| \le \frac{8 \max_{|z|=R} \Re(f(z) - f(0))}{R^n}.$$

PROOF. Let $c_n = a_n + ib_n$ for real a_n, b_n, and let $g(z) = f(z) - f(0)$. We have

$$\Re(g(Re^{i\theta})) = \Re\Big(\sum_{n=1}^{\infty} c_n R^n e^{in\theta}\Big) = \sum_{n=1}^{\infty} R^n a_n \cos(n\theta) - \sum_{n=1}^{\infty} R^n b_n \sin(n\theta).$$

Fourier inversion (or a simple calculation using uniform absolute convergence) then shows that

$$R^n a_n = \frac{1}{\pi} \int_0^{2\pi} \Re(g(Re^{i\theta})) \cos(n\theta) d\theta,$$
$$R^n b_n = -\frac{1}{\pi} \int_0^{2\pi} \Re(g(Re^{i\theta})) \sin(n\theta) d\theta.$$

Moreover, we see that $\int_0^{2\pi} \Re(g(Re^{i\theta})) d\theta = g(0) = 0$. Therefore

$$\begin{aligned} |c_n| \le 2\max(|a_n|, |b_n|) &\le \frac{2}{\pi R^n} \int_0^{2\pi} \Big|\Re(g(Re^{i\theta})\Big| d\theta \\ &= \frac{2}{\pi R^n} \int_0^{2\pi} \Big(\Big|\Re(g(Re^{i\theta})\Big| + \Re(g(Re^{i\theta}))\Big) d\theta \\ &\le \frac{8}{R^n} \max_{|z|=R} \Re(g(z)) = \frac{8}{R^n} \max_{|z|=R} \Re(f(z) - f(0)). \end{aligned}$$ □

LEMMA 9.2 (Partial fraction approximation for analytic functions). *Let $f(z)$ be an analytic function on the disk $|z| \le R$ with $f(0) \ne 0$. Let $z_1, \dots, z_k \in \mathbb{C}$ denote the zeros of f in the disk $|z| \le R/2$, listed with multiplicity. Then for $|z| \le 9R/20$ we have*

$$\Big|\frac{f'(z)}{f(z)} - \sum_{j=1}^{k} \frac{1}{z - z_j}\Big| \ll \frac{1}{R} \max_{|z|=R} \log\Big|\frac{f(z)}{f(0)}\Big|$$

PROOF. Let

$$g(z) = \frac{f(z)}{\prod_{j=1}^{k} (z - z_j)},$$

and $G(z) = \log(g(z)/g(0))$. Provided we are in a region where $G(z)$ is analytic, we have

$$\frac{f'(z)}{f(z)} - \sum_{j=1}^{k} \frac{1}{z - z_j} = \frac{\partial}{\partial z} G(z).$$

The function $g(z)$ has no zeros in the region $|z| \le R/2$, since we have removed all the zeros from f. Moreover, g is analytic in the region $|z| \le R$ since it only has removable singularities in this region and both the numerator and denominator are analytic. Thus the function $G(z) = \log(g(z)/g(0))$ is analytic in the region $|z| \le R/2$, and so has a Taylor expansion

$$G(z) = \sum_{n=1}^{\infty} c_n z^n$$

valid for $|z| \le R/2$. (Explicitly, we could define $G(z) := \int_1^z g'(s)ds/g(s)$ where the integral is a straight line contour from 1 to z.) By Lemma 9.1, we have that

$$|c_n| \le \frac{8}{(R/2)^n} \max_{|z|=R/2} \Re(G(z)) = \frac{8}{(R/2)^n} \max_{|z|=R/2} \log\Big|\frac{g(z)}{g(0)}\Big|$$
$$= \frac{8}{(R/2)^n} \log\Big(\max_{|z|=R/2}\Big|\frac{g(z)}{g(0)}\Big|\Big).$$

By the maximum modulus principle

$$\max_{|z|=R/2}\Big|\frac{g(z)}{g(0)}\Big| \le \max_{|z|=R}\Big|\frac{g(z)}{g(0)}\Big| = \max_{|z|=R}\Big|\frac{f(z)}{f(0)} \prod_{j=1}^{k} \frac{z_j}{z - z_j}\Big|.$$

Since $|z_j| \le R/2$ for all j, we have $|z_j|/|z - z_j| \le 1$ for $|z| = R$. Thus

$$|c_n| \le \frac{8}{(R/2)^n} \max_{|z|=R} \log\Big|\frac{f(z)}{f(0)}\Big|.$$

But now we have for $|z| \le 9R/20$

$$\Big|\frac{f'(z)}{f(z)} - \sum_{j=1}^{k} \frac{1}{z - z_j}\Big| = |G'(z)| = \Big|\sum_{n=1}^{\infty} c_n n z^{n-1}\Big|$$
$$\le \frac{16}{R}\Big(\max_{|z|=R} \log\Big|\frac{f(z)}{f(0)}\Big|\Big) \sum_{n=1}^{\infty} n\Big(\frac{9}{10}\Big)^{n-1}$$
$$\ll \frac{1}{R} \max_{|z|=R} \log\Big|\frac{f(z)}{f(0)}\Big|. \qquad \square$$

LEMMA 9.3 (Size of analytic function controls density of zeros). *Let $f(z)$ be an analytic function on the disk $|z| \le R$ then the number of zeros of f in the disk $|z| < R/2$ is bounded by*

$$2 \max_{|z|=R} \log\Big|\frac{f(z)}{f(0)}\Big|.$$

PROOF. If $|z| = R$ then

$$|z - z_\ell| = |\overline{z} - \overline{z_\ell}| = \left|\frac{z\overline{z} - z\overline{z_\ell}}{z}\right| = \left|\frac{R^2 - z\overline{z_\ell}}{R}\right|.$$

Thus if $f(z)$ has zeros $z_1, \dots, z_k$ in the disk $|z| < R/2$, then the function

$$h(z) = f(z) \prod_{j=1}^{k} \Big(\frac{R^2 - z\overline{z_j}}{R(z - z_j)}\Big)$$

is analytic on $|z| \le R$ and has $|h(z)| = |f(z)|$ if $|z| = R$. Therefore, by the maximum modulus principle

$$\max_{|z|=R} |f(z)| = \max_{|z|=R} |h(z)| \ge |h(0)| = |f(0)| \prod_{j=1}^{k} \frac{R}{|z_j|} \ge |f(0)| 2^k.$$

Thus the number of zeros k satisfies

$$k \le \frac{1}{\log 2} \max_{|z|=R} \log \left|\frac{f(z)}{f(0)}\right|. \qquad \square$$

LEMMA 9.4 (Zeros of $\zeta(s)$ are not too dense). *The number of non-trivial zeros ρ of $\zeta(s)$ with $t \le \Im(\rho) \le t + 1$ is $O(\log(2 + |t|))$.*

PROOF. The result is trivial for $t \le 10$ since $\zeta(s)$ can have only finitely many zeros in the region $\Re(s) \in [0, 1]$, $|\Im(s)| \le 10$. Therefore let $|t| \ge 10$. Let $g(z) = \zeta(1 + 1/100 + z + it)$ and $R = 3$, so $g(z)$ is analytic in $|z| < R$. By Lemma 4.5 we have $|g(z)| \ll t^3$ for $|z| = R$, and for all t we have

$$|g(0)| = |\zeta(1 + 1/100 + it)| \ge \prod_p \Big(1 + \frac{1}{p^{1+1/100}}\Big)^{-1} > 0.$$

Therefore, by Lemma 9.3, $g(z)$ can have $O(\log |t|)$ zeros with $|z| < 3/2$. But this means that $\zeta(s)$ can have at most $O(\log |t|)$ zeros with $\Im(\rho) \in [t - 1/2, t + 1/2]$, since these are all zeros of $g(z)$ with $|z| < R$. This gives the result. $\square$

LEMMA 9.5 (Partial fraction expansion of $\zeta(s)$). *Let $s = \sigma + it$ with $\sigma \ge -1/4$. Then we have*

$$\frac{\zeta'}{\zeta}(s) = \frac{-1}{s-1} + \sum_{|\rho - s| \le 1/10} \frac{1}{s - \rho} + O(\log(|t| + 2)).$$

Here the sum is over zeros ρ of $\zeta(s)$ with each zero of multiplicity m occurring m times.

PROOF. Again, the result is trivial for $|t| < 10$, since in this region $\zeta'(s)/\zeta(s)$ is $O(1)$ unless it is close to one of the finite number of poles, in which case $\zeta'(s)/\zeta(s) = -1/(s-1) + O(1)$ if s is close to 1, or $\zeta'(s)/\zeta(s) = 1/(s - \rho) + O(1)$ if s is close to a zero ρ. Similarly the result is trivial if $\Re(s) \ge 2$, since then $\zeta'/\zeta(s)$ is $O(1)$.

Therefore let $|t| \geq 10$. Let $g(z) = \zeta(1 + 1/100 + z + it)$ and $R = 3$ again, so $g(0) \gg 1$ uniformly in t and $|g(z)| \ll 1 + t^3$ for $|z| \leq R$. We see that the zeros of $g(z)$ with $|z| \leq R/2$ are of the form $\rho - 1 - 1/100 - it$ for zeros ρ of $\zeta(s)$ with $|\rho - 1 - 1/100 - it| \leq 3/2$. Now by Lemma 9.2 we have for $|z| < 27/20$

$$\frac{g'(z)}{g(z)} = \sum_{|\rho-1-1/100-it|\leq 3/2} \frac{1}{z - \rho + 1 + 1/100 + it} + O(\log(2 + |t|)).$$

We let $z = \sigma - 1 - 1/100$, so $g(z) = \zeta(\sigma + it)$, and note that $|z| < 27/20$ if $\sigma \geq -1/4$.

$$\frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)} = \sum_{|\rho-1-1/100-it|\leq 3/2} \frac{1}{\sigma + it - \rho} + O(\log(2 + |t|)).$$

We see that the set of ρ with $|\rho - 1 - 1/100 - it| \leq 3/2$ contains all ρ with $|\rho - \sigma - it| \leq 1/10$ since $\sigma \geq -1/4$. Since there are $O(\log t)$ zeros in the sum and any zero with $|\rho - \sigma - it| \geq 1/10$ contributes $O(1)$, we see that

$$\sum_{|\rho-1-1/100-it|\leq 3/2} \frac{1}{\sigma + it - \rho} = \sum_{|\rho-\sigma-it|\leq 1/10} \frac{1}{\sigma + it - \rho} + O(\log t).$$

Substituting this in above gives the result.

Those zeros in the sum with $|\rho - s| \geq 1/10$ can be absorbed into the error term by Lemma 9.4. This gives the result. □

Corollary 9.6 (Size of $\zeta'/\zeta(s)$ controlled away from zeros). *Let $s = \sigma + it$ with $\sigma \geq -1/4$. If s is a distance at least $\gg 1/\log(2 + |t|)$ from all zeros of $\zeta(s)$ and from 1 then*

$$\frac{\zeta'}{\zeta}(s) = O(\log(2 + |t|)^2).$$

Proof. This follows immediately from Lemma 9.4 and Lemma 9.5. □

CHAPTER 10

The Explicit Formula

LEMMA 10.1 (Computation of residues). *We have for* $\Re(s_0) \geq -1$

$$\operatorname*{Res}_{s=s_0} \frac{\zeta'}{\zeta}(s)\frac{x^s}{s} = \begin{cases} -x, & s_0 = 1, \\ m\dfrac{x^{s_0}}{s_0}, & s_0 \text{ a zero of zeta with multiplicity } m, \\ \frac{\zeta'}{\zeta}(0), & s_0 = 0, \\ 0, & \text{otherwise.} \end{cases}$$

PROOF. The function x^s has no poles in the complex plane, and the function $1/s$ has a simple pole at 0 and no other poles. By Lemma 7.8, $\zeta'/\zeta(s)$ has a simple pole at $s = 1$ and a simple pole at each zero of $\zeta(s)$. Thus the function

$$\frac{x^s}{s}\frac{\zeta'}{\zeta}(s)$$

has a simple pole at $s = 1$, a simple pole at $s = \rho$ for each zero of $\zeta(s)$, and a simple pole at $s = 0$. We want to calculate the residues at each of these poles, and this is easy since they are all simple poles. The residue at $s = 1$ is

$$\lim_{s\to 1}(s-1)\frac{\zeta'}{\zeta}(s)\frac{x^s}{s} = -x.$$

If ζ has a zero ρ of multiplicity m_ρ, the residue at $s = \rho$ is

$$\lim_{s\to \rho}(s-\rho)\frac{\zeta'}{\zeta}(s)\frac{x^s}{s} = m_\rho\frac{x^\rho}{\rho}.$$

Finally, the residue at $s = 0$ is

$$\lim_{s\to 0} x^s\frac{\zeta'}{\zeta}(s) = \frac{\zeta'}{\zeta}(0)$$

which is just some constant (in fact, it is equal to $\log 2\pi$). □

THEOREM 10.2 (The explicit formula). *Let* $2 \leq T \leq 2x$. *Then we have*

$$\psi(x) = x - \sum_{|\Im(\rho)|\leq T}\frac{x^\rho}{\rho} + O\Big(\frac{x(\log x)^3}{T}\Big).$$

Here the summation is over all non-trivial zeros of $\zeta(s)$*, occurring with multiplicity.*

PROOF. By Lemma 8.3, for any choice of $2 \le T_1 \le 2x$, we have

$$\psi(x) = \frac{-1}{2\pi i}\int_{c-iT_1}^{c+iT_1} x^s \frac{\zeta'}{\zeta}(s)\frac{ds}{s} + O\Big(\frac{x(\log x)^3}{T_1}\Big).$$

where $c = 1 + 1/\log x$. We want to estimate this integral using Cauchy's residue theorem applied to the box with corners $c - iT_1$, $c + iT_1$, $-1/4 + iT_1$ and $-1/4 - iT_1$. This gives

$$2\pi i \sum_{s_0} \operatorname*{Res}_{s=s_0} x^s \frac{\zeta'}{\zeta}(s)\frac{ds}{s} = \int_{c-iT_1}^{c+iT_1} x^s \frac{\zeta'}{\zeta}(s)\frac{ds}{s} + \int_{c+iT_1}^{-1/4+iT_1} x^s \frac{\zeta'}{\zeta}(s)\frac{ds}{s} + \int_{-1/4+iT_1}^{-1/4-iT_1} x^s \frac{\zeta'}{\zeta}(s)\frac{ds}{s} + \int_{-1/4-iT_1}^{c-iT_1} x^s \frac{\zeta'}{\zeta}(s)\frac{ds}{s},$$

provided that all these straight line contours avoid any poles, and where the sum over s_0 is over all poles of $x^s \frac{\zeta'}{\zeta}(s)/s$ in the box. We choose T_1 so that $T_1 \approx T$ and the horizontal contours stay away from the zeros of $\zeta(s)$. By Lemma 9.4 there are $O(\log T)$ zeros of $\zeta(s)$ with imaginary part between T and $T+1$ or between $-T$ and $-T-1$. Therefore there is some $T_1 \in [T, T+1]$ such that all zeros of $\zeta(s)$ satisfy

$$||\Im(\rho)| - T_1| \gg \frac{1}{\log T_1}.$$

Thus, with this choice of T_1, by Corollary 9.6, along the second, third and fourth integrals above we have

$$\frac{\zeta'}{\zeta}(s) = O(\log T)^2.$$

Therefore, bounding the integrand by its absolute value

$$\int_{c+iT_1}^{-1/4+iT_1} x^s \frac{\zeta'}{\zeta}(s)\frac{ds}{s} \ll \frac{x^c(\log T)^2}{T} \ll \frac{x(\log x)^2}{T},$$

and we get exactly the same bound for the integral between $-1/4 - iT_1$ and $c - iT_1$. For the integral between $-1/4 + iT_1$ and $-1/4 - iT_1$ we have

$$\int_{-1/4+iT_1}^{-1/4-iT_1} x^s \frac{\zeta'}{\zeta}(s)\frac{ds}{s} \ll \frac{(\log T)^2}{x^{1/4}} \int_{-T_1}^{T_1} \frac{dt}{1+|t|} \ll \frac{(\log T)^3}{x^{1/4}}.$$

By Lemma 10.1 we have

$$\sum_{s_0} \operatorname*{Res}_{s=s_0} x^s \frac{\zeta'}{\zeta}(s)\frac{ds}{s} = -x + \sum_{\substack{\rho \\ |\Im(\rho)| \le T_1}} \frac{x^\rho}{\rho}$$

where the sum is over all non-trivial zeros of $\zeta(s)$ appearing with multiplicity. Thus we find that

$$\frac{1}{2\pi i}\int_{c-iT_1}^{c+iT_1} x^s \frac{\zeta'}{\zeta}(s)\frac{ds}{s} = x - \sum_{\substack{\rho \\ |\Im(\rho)| \le T_1}} \frac{x^\rho}{\rho} + O\Big(\frac{(\log T)^3}{x^{1/4}} + \frac{x(\log T)^3}{T}\Big).$$

Finally, we see that there are $O(\log T)$ zeros ρ with $T \leq |\Im(\rho)| \leq T_1$ and they each contribute $O(x/T)$. Thus we find

$$\begin{aligned}\psi(x) &= \frac{-1}{2\pi i}\int_{c-iT_1}^{c+iT_1} x^s \frac{\zeta'}{\zeta}(s)\frac{ds}{s} + O\Big(\frac{x(\log x)^3}{T}\Big)\\ &= x - \sum_{\substack{\rho \\ |\Im(\rho)|\leq T_1}} \frac{x^\rho}{\rho} + O\Big(\frac{(\log T)^3}{x^{1/4}} + \frac{x(\log x)^3}{T}\Big)\\ &= x - \sum_{\substack{\rho \\ |\Im(\rho)|\leq T}} \frac{x^\rho}{\rho} + O\Big(\frac{x(\log x)^3}{T}\Big).\end{aligned}$$

In the final line we used the fact that $x(\log x)^3/T \gg (\log t)^3/x^{1/4}$ since $2 \leq T \leq 2x$. This gives the result. $\square$

Corollary 10.3 (Error term under RH). *Assume that all non-trivial zeros ρ of $\zeta(s)$ have $\Re(\rho) = 1/2$. Then we have*

$$\#\{p < x\} = \pi(x) = \int_2^x \frac{dt}{\log t} + O(x^{1/2}\log x).$$

Proof. Apply Theorem 10.2 with $T = x$. This gives

$$\psi(x) = x - \sum_{\substack{\rho \\ |\Im(\rho)|\leq x}} \frac{x^\rho}{\rho} + O(\log x)^3.$$

There are $O(\log(j+1))$ zeros with $|\Im(\rho)| \in [j, j+1]$ by Lemma 9.4, and if the real parts are all equal to $1/2$ then each one contributes $O(x^{1/2}/(j+1))$ to the sum above. Thus we have

$$\psi(x) = x + O\Big(x^{1/2}\sum_{0\leq j\leq x}\frac{\log(j+1)}{j+1} + (\log x)^3\Big) = x + O(x^{1/2}(\log x)^2).$$

Thus by Lemma 3.4, we have

$$\theta(x) = \psi(x) + O(x^{1/2}\log x) = x + O(x^{1/2}(\log x)^2).$$

Now Lemma 2.3 gives the result. $\square$

CHAPTER 11

The Prime Number Theorem

Proof idea. In the explicit formula, the contribution from zeros is small unless there are some zeros very close to the line $\Re(s) = 1$. If there is a zero $\rho_0 = \beta_0 + i\gamma_0$ with β_0 very close to 1, then by continuity we expect $\zeta(\sigma + i\gamma_0) \approx 0$ even when σ is slightly larger than 1. In this region $1/\zeta(\sigma+i\gamma_0) = \prod_p (1 - p^{-\sigma - i\gamma_0})$, which must be very large. But we would guess that this should only happen if $p^{i\gamma_0} \approx -1$ for many primes p so as to make the individual terms in the product large. But then $p^{2i\gamma_0} \approx 1$ for many primes p, and so $\zeta(\sigma + 2i\gamma_0) = \prod_p (1 - p^{-\sigma - 2i\gamma_0})^{-1}$ must be very large. But we know that $\zeta(s)$ has no poles in $s > 1$ and can't grow too much, which means that this cannot be the case. □

Lemma 11.1 (Size of ζ'/ζ at $\sigma + it$ controlled by size at $\sigma + 2it$). *Let $\sigma > 1$. Then we have*

$$4\Re\Big(\frac{\zeta'}{\zeta}(\sigma + it)\Big) \le -\Re\Big(\frac{\zeta'}{\zeta}(\sigma + 2it)\Big) - 3\frac{\zeta'}{\zeta}(\sigma).$$

Proof. We see that $|p^{it}| = 1$, and so using the triangle inequality

$$|1 - p^{2it}|^2 = |p^{it} - p^{-it}|^2 \le 2|p^{-it} + 1|^2 + 2|-1 - p^{-it}|^2 = 4|p^{it} + 1|^2. \tag{11.1}$$

We recall that for $\sigma > 1$

$$\frac{\zeta'}{\zeta}(\sigma + it) = -\sum_{n\ge1} \frac{\Lambda(n)}{n^{\sigma+it}} = -\sum_{m\ge1}\sum_p \frac{\log p}{p^{m\sigma + imt}}.$$

Since $|1 - p^{it}|^2 = 2(1 - \Re(p^{it}))$, we have

$$\sum_{m\ge1}\sum_p \frac{\log p}{p^{m\sigma}}\Big|1 - \frac{1}{p^{2imt}}\Big|^2 = 2\Re\Bigg(\sum_{m\ge1}\sum_p \frac{\log p}{p^{m\sigma}}\Big(1 - \frac{1}{p^{2imt}}\Big)\Bigg)$$
$$= -2\frac{\zeta'}{\zeta}(\sigma) + 2\Re\Big(\frac{\zeta'}{\zeta}(\sigma + 2it)\Big),$$

and similarly since $|1 + p^{it}|^2 = 2(1 + \Re(p^{it}))$, we have

$$\sum_{m\ge1}\sum_p \frac{\log p}{p^{m\sigma}}\Big|1 + \frac{1}{p^{imt}}\Big|^2 = -2\frac{\zeta'}{\zeta}(\sigma) - 2\Re\Big(\frac{\zeta'}{\zeta}(\sigma + it)\Big).$$

Thus the inequality (11.1) gives

$$-2\frac{\zeta'}{\zeta}(\sigma)+2\Re\Big(\frac{\zeta'}{\zeta}(\sigma+2it)\Big)\le -8\frac{\zeta'}{\zeta}(\sigma)-8\Re\Big(\frac{\zeta'}{\zeta}(\sigma+it)\Big),$$

which rearranges to give the result. □

Lemma 11.2 (Large real parts near zeros). *Let $\rho_0=\beta_0+i\gamma_0$ be a zero of $\zeta(s)$. Then we have for $\sigma>1$*

$$\Re\Big(\frac{\zeta'}{\zeta}(\sigma+i\gamma_0)\Big)\ge\frac{1}{\sigma-\beta_0}-O(\log(2+|\gamma_0|)),$$
$$\Re\Big(\frac{\zeta'}{\zeta}(\sigma+2i\gamma_0)\Big)\ge -O(\log(2+|\gamma_0|)).$$

Proof. Since $\zeta(s)$ is non-zero on the real line, any zero $\beta_0+i\gamma_0$ must have $|\gamma_0|\gg 1$. Then, by Lemma 9.5

$$\Re\Big(\frac{\zeta'}{\zeta}(\sigma+i\gamma_0)\Big)=\sum_{|\rho-\sigma-i\gamma_0|\le 1/10}\Re\Big(\frac{1}{\sigma+i\gamma_0-\rho}\Big)+O(\log(|\gamma_0|+2))$$
$$=\sum_{\substack{\rho=\beta+i\gamma\\ |\rho-\sigma-i\gamma_0|\le 1/10}}\frac{\sigma-\beta}{(\sigma-\beta)^2+(\gamma-\gamma_0)^2}+O(\log(|\gamma_0|+2)).$$

Since $\sigma>1\ge\Re(\rho)$ for all zeros ρ, we see that all terms in the sum contribute a positive quantity, and so we can drop all but the zero ρ_0 for a lower bound. This gives

$$\Re\Big(\frac{\zeta'}{\zeta}(\sigma+i\gamma_0)\Big)\ge\frac{1}{\sigma-\beta_0}-O(\log(2+|\gamma_0|)).$$

Similarly, we can drop all terms in the corresponding sum for $\Re(\zeta'/\zeta(\sigma+2i\gamma_0))$ for a lower bound, giving

$$\Re\Big(\frac{\zeta'}{\zeta}(\sigma+2i\gamma_0)\Big)\ge -O(\log(2+|\gamma_0|)).$$ □

Theorem 11.3 (The zero free region). *There is a constant $c>0$ such that if $\zeta(\sigma+it)=0$ then*

$$\sigma\le 1-\frac{c}{\log(2+|t|)}.$$

Proof. Assume for a contradiction that there is a zero $\rho_0=\beta_0+i\gamma_0$ with β_0 very close to 1. By Lemma 11.2 we have for $\sigma>1$

$$8\Re\Big(\frac{\zeta'}{\zeta}(\sigma+i\gamma_0)\Big)+2\Re\Big(\frac{\zeta'}{\zeta}(\sigma+2i\gamma_0)\Big)\ge\frac{8}{\sigma-\beta_0}-O(\log(2+|\gamma_0|)).$$

On the other hand, $\zeta'/\zeta(\sigma)=-1/(\sigma-1)+O(1)$. Thus Lemma (11.1) gives

$$\frac{8}{\sigma-\beta_0}\le\frac{6}{\sigma-1}+O(\log(2+|\gamma_0|)).$$

We now set $\sigma = 1 + \delta/\log(|\gamma_0| + 2)$. If δ is chosen to be a small enough constant then the right hand side is less than $7/(\sigma - 1)$. Thus we have

$$\frac{8}{\sigma - \beta_0} \leq \frac{7}{\sigma - 1},$$

which rearranges to

$$\beta_0 \leq \frac{8 - \sigma}{7} = 1 - \frac{\delta}{7\log(2 + |\gamma_0|)}.$$

This gives the result. $\square$

THEOREM 11.4 (The Prime Number Theorem). *There is a constant $c > 0$ such that*

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O\Big(x \exp(-c\sqrt{\log x})\Big).$$

In particular,

$$\pi(x) = \frac{x}{\log x} + o\Big(\frac{x}{\log x}\Big).$$

PROOF. We apply Lemma 10.2 to find that for $2 \leq T \leq x$

$$\psi(x) = x - \sum_{|\Im(\rho)| \leq T} \frac{x^\rho}{\rho} + O\Big(\frac{x(\log x)^3}{T}\Big).$$

By Theorem 11.3 for each term ρ in the sum we have

$$|x^\rho| = x^{\Re(\rho)} \leq x^{1 - c/\log T},$$

for some suitable constant $c > 0$. Thus

$$\psi(x) = x + O\Big(x^{1-c/\log T} \sum_{|\Im(\rho)| \leq T} \frac{1}{|\rho|}\Big) + O\Big(\frac{x(\log x)^3}{T}\Big).$$

Since there are $O(\log(1 + j))$ zeros with $|\Im(\rho)| \in [j, j + 1]$, we see that the sum is of size $O(\log T)^2$. Thus we have

$$\psi(x) = x + O\Big(x^{1-c/\log T}(\log T)^2\Big) + O\Big(\frac{x(\log x)^3}{T}\Big).$$

We now choose $T = \exp(\sqrt{\log x})$ to balance the size of the two error terms. Thus, for a suitable constant c'

$$\psi(x) = x + O\Big(x \exp(-c'\sqrt{\log x})\Big).$$

We recall from Lemma 2.3 and Lemma 3.4 that

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t \log^2 t} dt \qquad \text{and} \qquad \theta(x) = \psi(x) + O(x^{1/2} \log x).$$

Thus $\theta(t) = t + O(t\exp(-c'\sqrt{\log t}))$, and so

$$\begin{aligned}\pi(x) &= \frac{x + O(x\exp(-c'\sqrt{\log x}))}{\log x} + \int_2^x \frac{1 + O(\exp(-c'\sqrt{\log t}))}{\log^2 t}dt \\ &= \int_2^x \frac{dt}{\log t} + O\Big(x\exp(-c'\sqrt{\log x})\Big).\end{aligned}$$

□

CHAPTER 12

Primes in Short Intervals

We will use the following two facts, whose proofs are more involved and we will not give here.

FACT 12.1 (Improved zero free region). *There is a constant $c > 0$ such that if $\zeta(\sigma + it) = 0$ then*

$$\sigma \leq 1 - \frac{c}{\log(2 + |t|)^{2/3} \log\log(3 + |t|)^{1/3}}.$$

FACT 12.2 (Zero Density Estimate). *Let $N(\sigma, T)$ denote the number of zeros ρ of $\zeta(s)$ such that $\Re(\rho) \geq \sigma$ and $|\Im(\rho)| \leq T$. Then, for any $\epsilon > 0$ there is a constant $C(\epsilon) > 0$ such that for $T \geq 1$ and $\sigma \geq 1/2$*

$$N(\sigma, T) \leq C(\epsilon) T^{\frac{12+\epsilon}{5}(1-\sigma)}.$$

THEOREM 12.3 (Primes in short intervals). *Let $\epsilon > 0$ and x be large enough in terms of ϵ. Then for $x^{7/12+\epsilon} \leq y \leq x$ we have*

$$\#\{p \in [x, x+y]\} = \frac{y}{\log x} + o\Big(\frac{y}{\log x}\Big).$$

PROOF. By the Explicit Formula (Theorem 10.2), we have for $2 \leq T \leq x$ and for $y \leq x$

$$\psi(x+y) = x + y - \sum_{|\Im(\rho)| \leq T} \frac{(x+y)^\rho}{\rho} + O\Big(\frac{x(\log x)^3}{T}\Big),$$

$$\psi(x) = x - \sum_{|\Im(\rho)| \leq T} \frac{x^\rho}{\rho} + O\Big(\frac{x(\log x)^3}{T}\Big).$$

Thus

$$\psi(x+y) - \psi(x) = y + \sum_{|\Im(\rho)| \leq T} O\Big(\Big|\frac{(x+y)^\rho - x^\rho}{\rho}\Big|\Big) + O\Big(\frac{x(\log x)^3}{T}\Big).$$

Consider the contribution of zeros which have

$$\sigma_1 \leq \Re(\rho) \leq \sigma_1 + \frac{1}{\log x}$$

for some $1/2 \leq \sigma_1 \leq 1$. We see that for each such zero

$$\Big|\frac{(x+y)^\rho - x^\rho}{\rho}\Big| = \Big|\int_x^{x+y} t^{\rho-1} dt\Big| \leq \int_x^{x+y} t^{\Re(\rho)-1} dt \ll y x^{\sigma_1 - 1}.$$

By Fact 12.1, there are no such zeros unless $\sigma_1 \le 1 - c'/(\log x)^{4/5}$ for some constant $c' > 0$ (recall $T \le x$). If this is the case, then by Fact 12.2 there are $O(T^{(12/5+\epsilon)(1-\sigma_1)})$ such zeros. Thus the total contribution is

$$\ll T^{(12/5+\epsilon)(1-\sigma_1)} y x^{\sigma_1 - 1} \ll y\Big(\frac{T^{12/5+\epsilon}}{x}\Big)^{1-\sigma_1}.$$

If $T^{12/5+\epsilon} \le x^{1-\epsilon}$ then since $1 - \sigma_1 \ge c'/(\log x)^{4/5}$, this is

$$\ll y(x^{\epsilon})^{-c'/(\log x)^{4/5}} \ll y \exp\Big(-c'\epsilon(\log x)^{1/5}\Big).$$

All zeros ρ with $\Re(\rho) \ge 1/2$ satify $j/\log x \le \Re(\rho) \le (j+1)/\log x$ for some integer $j \le \log x$. Thus we see that all zeros with $\Re(\rho) \ge 1/2$ appearing in the sum contribute a total

$$\ll \log(x) \cdot y \exp\Big(-c'\epsilon(\log x)^{1/5}\Big) \ll \frac{y}{\log x}.$$

The zeros with $\Re(\rho) \le 1/2$ contribute $O(x^{1/2}(\log x)^2)$ in total, as in Corollary 10.3. Thus if $T^{12/5+\epsilon} \le x^{1-\epsilon}$ we find

$$\psi(x+y) - \psi(x) = y + O\Big(\frac{y}{\log x}\Big) + O(x^{1/2}\log x) + O\Big(\frac{x(\log x)^3}{T}\Big).$$

If we choose T such that $T^{12/5+\epsilon} = x^{1-\epsilon}$, this simplifies to

$$\psi(x+y) - \psi(x) = y + o(y) + o(x^{7/12+\epsilon}).$$

In particular for $y \ge x^{7/12+\epsilon}$, this gives $\psi(x+y) - \psi(x) = y + o(y)$. Partial summation then gives the result. □

Theorem 12.4 (Primes in almost all short intervals). *Let $\epsilon > 0$ and $2 \le x$ and $1 \ge \delta \ge x^{-5/6+\epsilon}$. Then for all but $o(x)$ values of $t \le x$ we have*

$$\#\{p \in [t, t+\delta t]\} = \frac{\delta t}{\log t} + o\Big(\frac{\delta t}{\log t}\Big).$$

Proof. By parital summation, it suffices to show that for all but $o(x)$ values of $t \le x$ we have

$$\psi(t+\delta t) - \psi(t) = \delta t + o(\delta t).$$

Imagine for a contradiction that there is a constant $\epsilon > 0$ such that the set $\mathcal{S} \subset [0,x]$ for which $|\psi(t+\delta t) - \psi(t) - \delta t| \ge \epsilon\delta t$ has measure $\ge \epsilon x$. Then we see that

$$\int_0^x \Big|\psi(t+\delta t) - \psi(t) - \delta t\Big|^2 dt \ge \int_{\mathcal{S}} \Big|\psi(t+\delta t) - \psi(t) - \delta t\Big|^2 dt \gg \epsilon^3\delta^2 x^3.$$

Therefore to get a contradiction it suffices to show that

$$\int_0^x \Big|\psi(t+\delta t) - \psi(t) - \delta t\Big|^2 dt = o(\delta^2 x^3).$$

As in the proof of Theorem 12.3, by the Explicit Formula we see that

$$\psi(t+\delta t)-\psi(t)-\delta t=\sum_{|\Im(\rho)|\le T}\frac{t^{\rho}\Big((1+\delta)^{\rho}-1\Big)}{\rho}+O\Big(\frac{t(\log t)^3}{T}\Big)$$

for any $2\le T\le x$. Since $(a+b)^2\ll a^2+b^2$, we find

$$\int_0^x\Big|\psi(t+\delta t)-\psi(t)-\delta t\Big|^2dt\ll\int_0^x\Big|\sum_{|\Im(\rho)|\le T}\frac{t^{\rho}\Big((1+\delta)^{\rho}-1\Big)}{\rho}\Big|^2dt+O\Big(\frac{x^3(\log x)^6}{T^2}\Big).$$

Expanding the sum and performing the integration gives

$$\begin{aligned}\int_0^x\Big|&\sum_{|\Im(\rho)|\le T}\frac{t^{\rho}\Big((1+\delta)^{\rho}-1\Big)}{\rho}\Big|^2dt\\&=\sum_{|\Im(\rho_1)|\le T}\sum_{|\Im(\rho_2)|\le T}\frac{\Big((1+\delta)^{\rho_1}-1\Big)\Big((1+\delta)^{\overline{\rho_2}}-1\Big)}{\rho_1\overline{\rho_2}}\int_0^x t^{\rho_1+\overline{\rho_2}}dt\\&=\sum_{|\Im(\rho_1)|\le T}\sum_{|\Im(\rho_2)|\le T}\frac{\Big((1+\delta)^{\rho_1}-1\Big)\Big((1+\delta)^{\overline{\rho_2}}-1\Big)}{\rho_1\overline{\rho_2}}\Big(\frac{x^{\rho_1+\overline{\rho_2}+1}}{\rho_1+\overline{\rho_2}+1}\Big).\end{aligned}$$

As in the proof of Theorem 12.3, we have that $|((1+\delta)^{\rho}-1)/\rho|\ll\delta$. We also see that $|x^{\rho_1+\overline{\rho_2}}|\ll x^{2\Re(\rho_1)}+x^{2\Re(\rho_2)}$. Thus, by symmetry we obtain the bound

$$\ll\delta^2\sum_{|\Im(\rho_1)|\le T}x^{2\Re(\rho_1)+1}\sum_{|\Im(\rho_2)|\le T}\frac{1}{|\rho_1+\overline{\rho_2}+1|}.$$

Since there are $O(\log(j+1))$ zeros ρ with $|\Im(\rho)|\in[j,j+1]$, the inner sum is $O(\log T)\ll\log x$. Thus we have shown that

$$\int_0^x\Big|\psi(t+\delta t)-\psi(t)-\delta t\Big|^2dt\ll\delta^2x\log x\sum_{|\Im(\rho)|\le T}x^{2\Re(\rho)}+\frac{x^3(\log x)^6}{T^2}.$$

We bound the inner sum in an analogous way to the proof of Theorem 12.3 by considering those zeros ρ with $\Re(\rho)\in[\sigma_1,\sigma_1+1/\log x]$ for some $\sigma_1\ge1/2$. There are no such zeros if $\sigma_1\ge1-c'/(\log x)^{4/5}$, and otherwise there are $O(T^{(12/5+\epsilon)(1-\sigma_1)})$ zeros. Each zero contributes $O(x^{2\sigma_1})$ to the sum, so the total contribution is

$$\ll x^2\Big(\frac{T^{12/5+\epsilon}}{x^2}\Big)^{1-\sigma_1}.$$

Thus, provided $T^{12/5+\epsilon}\le x^{2-\epsilon}$, we find that these zeros contribute $O(x^2\exp(-\epsilon c'(\log x)^{1/5})$. By considering $\sigma_1=j/\log x$ for an integer $j\le\log x$, we see that the total contribution from all of the zeros with $\Re(\rho)\ge1/2$ is

$$\ll(\log x)x^2\exp\Big(-c'\epsilon(\log x)^{1/5}\Big)=o\Big(\frac{x^2}{\log x}\Big).$$

The contribution from zeros with $\Re(\rho) \leq 1/2$ is $O(xT \log T) = o(x^2/\log x)$. Therefore, putting this together we see

$$\int_0^x \Big| \psi(t+\delta t) - \psi(t) - \delta t \Big|^2 dt = o(\delta^2 x^3),$$

as required. □