Question 1. Prove the following.

(i) $\log^4 X < X^{1/10}$ for all sufficiently large X;

(ii) $e^{\sqrt{\log X}} = O_\varepsilon(X^\varepsilon)$ for all $\varepsilon > 0$ and $X \geqslant 1$;

(Here $O_\epsilon(h(x))$ means a function $g(x)$ which satisfies $|g(x)| \leq C_\epsilon h(x)$ for some constant C_ϵ depending only on ϵ)

(iii) $X(1 + e^{-\sqrt{\log X}}) + X^{3/4} \sin X \sim X$.

Question 2. Let $\mathrm{Li}(x) := \int_2^x \frac{dt}{\log t}$.

(i) Show that $\mathrm{Li}(x) = \frac{x}{\log x} + O\Big(\frac{x}{(\log x)^2}\Big)$ for $x \geq 3$.

(ii) Show that for any $k \geq 1$, $\mathrm{Li}(x) = \sum_{j=1}^{k} \frac{(j-1)!x}{(\log x)^j} + O\Big(\frac{k!^2 x}{(\log x)^{k+1}}\Big)$ for $x \geq 3$.

Question 3. In the following exercise, $a(X), b(X) \geq 2$ are functions tending to ∞ as $X \to \infty$. For each statement below, either give a proof of its correctness or a counterexample.

(i) If $a(X) \sim b(X)$ then $\frac{a(X)}{\log(a(X))} \sim \frac{b(X)}{\log(b(X))}$.

(ii) If $a(X) - b(X) \to 0$ then $a(X) \sim b(X)$.

(iii) If $a(X) \sim b(X)$ then $a(X) - b(X) \to 0$.

(iv) If $a(X) \sim b(X)$ and $a'(X) := \sum_{y \leqslant X} a(y)$, $b'(X) := \sum_{y \leqslant X} b(y)$ then $a'(X) \sim b'(X)$.

(v) If $a'(X) \sim b'(X)$ where $a'(X) := \sum_{y \leqslant X} a(y)$, $b'(X) := \sum_{y \leqslant X} b(y)$, then $a(X) \sim b(X)$.

Question 4. Show that there are arbitrarily large gaps between consecutive primes by

(i) using the bound $\pi(x) = O(x/\log x)$;

(ii) considering the numbers $n! + 2, \ldots, n! + n$.

Which of the two approaches gives the better bound?

Question 5. Assume that $\pi(x) \sim x/\log x$.

(i) Show that $p_n \sim n \log n$, where p_n denotes the nth prime.

(ii) Deduce that $p_{n+1} \sim p_n$ and $\sup_{p_n \leq x}(p_{n+1} - p_n) = o(x)$.

Question 6. Let X be an integer.

(i) Show that
$$\log(X!) = \sum_{n\le X} \log(n),$$
and
$$\log(X!) = \sum_{p\le X} \log p\Big(\lfloor\frac{X}{p}\rfloor + \lfloor\frac{X}{p^2}\rfloor + \dots\Big).$$

(ii) Show that
$$\sum_{n\leqslant X} \log n = X\log X - X + O(\log X),$$
and so
$$\sum_{p\leqslant X} \log p\big(\lfloor\frac{X}{p}\rfloor + \lfloor\frac{X}{p^2}\rfloor + \dots\big) = X\log X - X + O(\log X).$$

(iii) Show that the contribution from the terms $\lfloor\frac{X}{p^k}\rfloor$ with $k \geqslant 2$ is $O(X)$.

(iv) Deduce Mertens' first estimate
$$\sum_{p\leqslant X} \frac{\log p}{p} = \log X + O(1).$$
Explain why this remains valid even if X is not necessarily an integer.

Question 7. Using Mertens' first estimate above, prove the second Mertens estimate: we have
$$\sum_{p\leqslant X} \frac{1}{p} = \log\log X + O(1).$$

Deduce that there are constants $c_1, c_2 > 0$ such that
$$\frac{c_1}{\log X} \leqslant \prod_{p\leqslant X}(1 - \frac{1}{p}) \leqslant \frac{c_2}{\log X}.$$

Question 8. Let p_n denote the nth prime.

(i) Is it the case that, for sufficiently large n, the sequence $p_{n+1} - p_n$ is strictly increasing?

(ii) Is it the case that, for sufficiently large n, the sequence $p_{n+1} - p_n$ is nondecreasing?

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