

Question 1. Prove that $\pi(X) \leq 2\pi(X/2)$ for X sufficiently large.

Question 2. Let p_n denote the n^{th} prime. Prove that

$$p_n = n \log n + n \log \log n + O(n).$$

Question 3. (i) Let $\theta \in (0, 1)$ be such that $\Re(\rho) \leq \theta$ for all non-trivial zeros ρ . Deduce that for all $x \geq 2$

$$\sum_{n < x} \Lambda(n) = x + O(x^\theta (\log x)^2).$$

(ii) Let $\gamma \in (0, 1)$ be such that for all $x \geq 2$

$$\sum_{n < x} \Lambda(n) = x + O(x^\gamma).$$

Show that $\Re(\rho) \leq \gamma$ for all zeros of $\zeta(s)$.

(Hint: Use partial summation to prove analytic continuation of ζ'/ζ)

(iii) Let $\alpha \in (0, 1)$ be fixed. Show that if for all $x \geq 2$ we have

$$\sum_{n < x} \Lambda(n) = x + O\left(x^\alpha \exp(\sqrt{\log x})\right)$$

then in fact

$$\sum_{n < x} \Lambda(n) = x + O(x^\alpha (\log x)^2).$$

Question 4. (i) Show that for $\Re(s) > 1$ we have

$$\log \zeta(s) = \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}}.$$

(ii) Show that $3 + 4 \cos(\theta) + \cos(2\theta) \geq 0$.

(iii) Using (i) and (ii), show that for $\sigma > 1$

$$3 \log \zeta(\sigma) + 4 \Re \log \zeta(\sigma + it) + \Re \log(\zeta(\sigma + 2it)) \geq 0.$$

Deduce from this that for $\sigma > 1$

$$\zeta(\sigma)^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| \geq 1.$$

(iv) Deduce from the above inequality that $\zeta(1 + it) \neq 0$.

(Hint: Consider $\sigma \rightarrow 1$)

Question 5. It is a fact that

$$\sum_{n < x} \Lambda(n) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log(1 - x^{-2}),$$

where the sum is understood to be the limit as $T \rightarrow \infty$ of $\sum_{|\Im(\rho)| \leq T} x^{\rho}/\rho$ over non-trivial zeros ρ (it is not absolutely convergent).

- (i) Using this fact, show that $\zeta(s)$ must have at least one non-trivial zero.
- (ii) Show that if ρ is a non-trivial zero of $\zeta(s)$, then so is $1 - \rho$.
- (iii) Let $\epsilon > 0$. Using Question 3, deduce that we cannot have for all $x \geq 2$ that

$$\sum_{n < x} \Lambda(n) = x + O(x^{1/2-\epsilon}).$$

Question 6. Recall from Sheet 3 Q6: If $f : \mathbb{R} \rightarrow \mathbb{C}$ is smooth and non-zero only on some interval $[\alpha, \beta] \subseteq (0, \infty)$ then for any $\sigma \in \mathbb{R}$

$$f\left(\frac{n}{y}\right) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F(s) \frac{y^s}{n^s} ds,$$

where $F(s) = \int_0^{\infty} f(x)x^{s-1}dx$ is a smooth function with $|F(\sigma + it)| \ll 1/|t|^{100}$ and with no zeros or poles.

- (i) Show that

$$\sum_{n=1}^{\infty} \Lambda(n) f\left(\frac{n}{y}\right) = \frac{-1}{2\pi i} \int_{2-i\infty}^{2+i\infty} y^s F(s) \frac{\zeta'(s)}{\zeta(s)} ds.$$

- (ii) Deduce that

$$\sum_{n=1}^{\infty} \Lambda(n) f\left(\frac{n}{y}\right) = y \int_0^{\infty} f(t) dt - \sum_{\rho} y^{\rho} F(\rho) + O(y^{-1/4})$$

where \sum_{ρ} is a sum over all non-trivial zeros of $\zeta(s)$ with multiplicity.

Question 7. For this question you may use the following fact: $|\zeta(\sigma + it)| \ll |t|^{1-\sigma} \log |t|$ for $\sigma \leq 1$ and $|t| \geq 1$.

- (i) Show using Perron's formula that for $2 \leq T \leq 2x$

$$\sum_{n < x} \frac{\mu(n)^2 \phi(n)}{n} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s} \zeta(s) Z(s) ds + O\left(\frac{x(\log x)^3}{T}\right),$$

where $c = 1 + 1/\log x$ and for $\Re(s) > 1$

$$Z(s) = \prod_p \left(1 - \frac{1}{p^{2s}} - \frac{1}{p^{s+1}} + \frac{1}{p^{2s+1}}\right).$$

(ii) Show that the product of for $Z(s)$ converges absolutely for $\Re(s) > 1/2$.

(iii) Let $\epsilon = 1/1000$. By moving the line of integration to $\Re(s) = 1/2 + \epsilon$, show that

$$\sum_{n < x} \frac{\mu(n)^2 \phi(n)}{n} = x \prod_p \left(1 - \frac{2}{p^2} + \frac{1}{p^3}\right) + O\left(x^{1/2+\epsilon} T^{1/2-\epsilon} \log x\right) + O\left(\frac{x(\log x)^3}{T}\right)$$

(iv) Deduce that

$$\sum_{n < x} \frac{\mu(n)^2 \phi(n)}{n} = x \prod_p \left(1 - \frac{2}{p^2} + \frac{1}{p^3}\right) + O\left(x^{2/3+\epsilon}\right).$$

Question 8. Recall from Sheet 3 Q6: If $f : \mathbb{R} \rightarrow \mathbb{C}$ is smooth and non-zero only on some interval $[\alpha, \beta] \subseteq (0, \infty)$ then for any $\sigma \in \mathbb{R}$

$$f\left(\frac{n}{y}\right) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F(s) \frac{y^s}{n^s} ds,$$

where $F(s) = \int_0^\infty f(x) x^{s-1} dx$ is a smooth function with $|F(\sigma + it)| \ll 1/|t|^{100}$.

(i) Show that

$$\sum_{n=1}^{\infty} \frac{\mu(n)^2 \phi(n)}{n} f\left(\frac{n}{y}\right) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} y^s F(s) \zeta(s) Z(s) ds$$

where $Z(s)$ is the function appearing in Question 7.

(ii) Fix $\epsilon > 0$. Show that

$$\sum_{n=1}^{\infty} \frac{\mu(n)^2 \phi(n)}{n} f\left(\frac{n}{y}\right) = y \left(\int_0^\infty f(x) dx\right) \prod_p \left(1 - \frac{2}{p^2} + \frac{1}{p^3}\right) + O(y^{1/2+\epsilon})$$

(Compare the answer here to that in Question 7)

Question 9 (Bonus Question). Let $\sigma(n) = \sum_{d|n} d$ be the sum of divisors of n . Following the approach of Question 7 or Question 8, obtain an asymptotic formula for $\sum_{n < x} \mu(n)^2 \sigma(n)$ or $\sum_n \mu(n)^2 \sigma(n) f(n/y)$.

Question 10 (Bonus Question). Let $\Psi(x, y) = \{n \leq x : p|n \Rightarrow p \leq y\}$ be the set of integers up to x which only involve prime factors of size at most y .

(i) Let $\alpha \in (0, 1)$ be fixed. Show that as $x \rightarrow \infty$

$$\sum_{x^\alpha \leq p \leq x} \frac{1}{p} = \log \frac{1}{\alpha} + o(1).$$

(ii) Show that for $x^{1/2} \leq y \leq x$ we have

$$\Psi(x, y) = \left(1 - \log\left(\frac{\log x}{\log y}\right) + o(1)\right)x.$$

(iii) Show that for any $x \geq 1$ and $z \geq y > 0$

$$\Psi(x, y) = \Psi(x, z) - \sum_{p < y \leq z} \Psi\left(\frac{x}{p}, p\right).$$

(iv) Deduce that for $x^{1/3} \leq y \leq x^{1/2}$

$$\Psi(x, y) = \left(1 - \log 2 - \int_2^{\log x / \log y} \frac{1}{v} \left(1 - \log(v-1)\right) dv + o(1)\right)x$$

(v) Define a function $\rho : [0, \infty) \rightarrow \mathbb{R}$ by $\rho(u) = 1$ if $u \leq 1$ and for $u > 1$

$$\rho(u) = 1 - \int_1^u \rho(t-1) \frac{dt}{t}.$$

Show that parts (ii) and (iv) imply that for $u \leq 3$

$$\Psi(x, x^{1/u}) = (\rho(u) + o(1))x.$$

(vi) Show by induction that if

$$\Psi(x, x^{1/u}) = (\rho(u) + o(1))x$$

for $u \leq m$, then the same equation holds for $u \leq m + 1$. Deduce that for any fixed $u > 0$ we have

$$\Psi(x, x^{1/u}) = (\rho(u) + o(1))x.$$

james.maynard@maths.ox.ac.uk